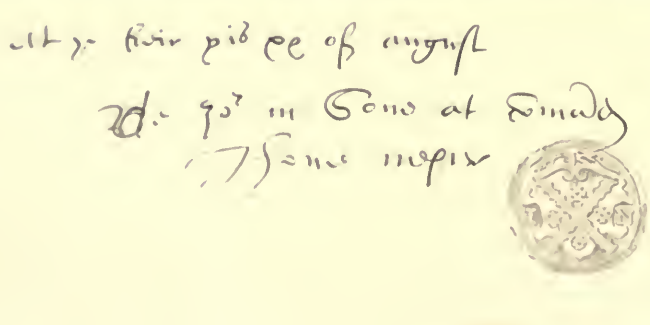
DE ARTE LOGISTICA.

[illegible] From the [illegible] in the [illegible] of the Lord Napier



DE ARTE LOGISTICA

JOANNIS NAPERI

MERCHISTONII BARONIS

LIBRI QUI SUPERSUNT.

IMPRESSUM EDINBURGI

M.DCCC.XXXIX.

At a Meeting of the COMMITTEE of the BANNATYNE CLUB, held in the House of the President, on Monday the 28th of January, 1839,—

" RESOLVED, That One Hundred and One Copies of a Volume, entitled JOANNIS NEPERI À MERCHISTOUN BARONIS DE ARTE LOGISTICA LIBRI TRES, be subscribed for in the name of the Club, to MARK NAPIER, ESQ., Advocate ; and that Club paper be furnished for that number."

Extracted from the Minutes of the Club.

DAVID LAING, *Secretary.*

THE BANNATYNE CLUB.

SEPTEMBER, M.DCCC.XXXIX.

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TO THE RIGHT HONOURABLE

FRANCIS LORD NAPIER OF MERCHISTON,

ETC. ETC. ETC.

My dear Lord Napier,

This Memorial of your great Ancestor, which it has been my ambition to present in a form worthy of the genius it records, I dedicate to you, in remembrance of the exemplary liberality with which the Archives of your noble family have been always open to illustrate the History and the Letters of Scotland.

Yours affectionately,

MARK NAPIER.

Edinburgh, *November* 1, 1839.

INTRODUCTION.

[illegible] by [illegible] M'Haffie. From the original sketch by H. W. Williams in the [illegible] of Mark Napier Esq.

INTRODUCTION.

In the Memoirs of Napier of Merchiston, published in 1834, some account was given of two manuscript treatises—one of Arithmetic, and the other of Algebra—composed in Latin by that celebrated mathematician, and which had remained inedited in the charter-chest of his family, and indeed unknown to the world, until the Memoirs were published. Upon that occasion, little more could be afforded than a very imperfect review of their contents. The idea subsequently occurred, that it might gratify the lovers of science if these mathematical studies of (to adopt the expressions of the historian Hume) "The celebrated Inventor of Logarithms, the person to whom the title of Great Man is more justly due than to any other whom his country ever produced,"—were added as an appendix

to a new edition of his Life. I have been induced, however, to publish the treatises in their present independent and more becoming form, by the spirited interposition of the Bannatyne and Maitland Clubs of Scotland; whose unanimous patronage of the work,—with their characteristic care for, and pride in, the ancient letters of Scotland,—has alone enabled me to render the volume so worthy a memorial of Scotish genius.

Napier's scientific manuscripts came into the possession, not of his eldest son Archibald, the first Lord Napier of Merchiston, but of a younger son Robert. From the congeniality of their pursuits, Robert became his father's literary executor, and edited his posthumous work, entitled "Logarithmorum Canonis Constructio," being his secret of the construction of that Canon of the Logarithms which he had published three years before his death. Robert Napier, designed of Bowhopple, Culcreugh, and Drumquhannie, was the second son of Napier's second marriage. The late Colonel Milliken Napier, Robert's lineal male representative, was still in possession of many private papers of the family at the close of last century. Upon one occasion, when the Colonel was called from home on the service of his country, these papers, along with a portrait of the great Napier, and a Bible with his autograph, were deposited for safety in a room of the house of Milliken, in

Renfrewshire. During the owner's absence the house was burned to the ground, and the precious relics perished.

As these manuscripts had not been arranged, nor properly examined, the extent of the loss is unknown; but it is more than probable that among them were the originals from which Napier's son Robert made the transcripts accidentally preserved, and now given to the public. The manner in which these transcripts had escaped the fate of the Culcreuch papers appears from the following note, written in the MS. volume itself, by Francis seventh Lord Napier, grandfather of the present Peer to whom it now belongs:—

"John Napier of Merchiston, inventor of the Logarithms, left his manuscripts to his son Robert, who appears to have caused the following pages to have been written out fair from his father's notes, for Mr Briggs, Professor of Geometry at Oxford. They were given to Francis, the fifth Lord Napier, by William Napier of Culcreuch, Esq., heir-male of the above-named Robert. Finding them, in a neglected state, amongst my family papers, I have bound them together, in order to preserve them entire.

"NAPIER."

"*7th March*, 1801."

The transcripts are entirely in the handwriting of Robert Napier himself, as is ascertained beyond doubt by a comparison with some of his letters in Lord Napier's charter-chest; and this title, written on the first leaf, is also in his handwriting :—

"The Baron of Merchiston his booke of Arithmeticke and Algebra. For Mr Henrie Briggs, Professor of Geometrie at Oxforde."

That they were taken directly from the author's own papers, is proved by the note which the transcriber adds at the conclusion of the fragment on Geometrical Logistic :—

"I could find no more of this Geometricall pairt amongst all his fragments."

And also by another note which immediately follows the abrupt termination of the Algebra :—

"There is no more of his Algebra orderlie sett doun."

These notes, of which fac-similes will be found in their respective places in this volume, seem to have been addressed to Henry Briggs; and they not only prove that the transcripts were made from Napier's private manuscripts, but that Napier himself had so far written out the treatises in the form now published, and that more remained to be digested and arranged.

The late Lord Napier, whose untimely death in China, a few years ago, attracted the mournful interest of the public, intrusted to myself, before his departure on that mission, the many curious and valuable historical relics of his family, with full permission to publish whatever might serve to illustrate the history and letters of Scotland. This task I have now accomplished, with at least industrious zeal, in the Memoirs of Merchiston, the History of the Lennox, the Memoirs of Montrose, and, finally, the editing of this beautiful, though unfinished treatise, De Arte Logistica.

The authenticity of the MS., as an unpublished work of the Inventor of Logarithms, being thus unquestionable, the date of its composition becomes an interesting enquiry, in reference to the history of science in Scotland. As Napier died in the year 1617, no later date can be assigned to these mathematical studies than the commencement of the seventeenth century—an epoch when the light of science is only said to have dawned, in our country, because of the publication of his own invention of Logarithms. From the internal evidence, however, there is every reason to suppose, that, when so much of his Arithmetic and Algebra was thus "orderlie sett doun" by Napier, the system of Logarithms was not only unknown to the world, but had not as yet been developed in the mind of the inventor

himself. Consequently, a much earlier date must be assigned to the compositions now published. They are, in all probability, the first-fruits of that immortal genius which eventually drew from the mysterious depths of Numbers their pearl of greatest price. Napier's Arithmetic and Algebra will be regarded with reverential interest by all who can, in any degree, appreciate the analytical power, and the practical effects of the Logarithms; especially if, in those treatises, we can trace the preparatory labours of this secluded Scotish Baron, ere he so unexpectedly displayed himself on a pinnacle of science to which Kepler himself paid homage.

Napier published his great work, entitled "Mirifici Canonis Logarithmorum Descriptio," in the year 1614. By that time his mind was so deeply engrossed with the subject, and he was so constantly employed,—during the short interval between the date of that publication and his death in 1617,—with his plans for extricating science from the trammels of ordinary calculation, that his Arithmetic and Algebra can hardly be imagined to have been his latest compositions. But what appears decisive upon the point is, that there cannot be discovered throughout the whole of the work now printed an allusion even to the principle of the Logarithms. A complete exposition of the subject was not, indeed, to be expected in his

treatise of Arithmetic; but neither is it to be supposed that he would have written, for instance, the fifth chapter of his second book of Arithmetic, "De Multiplicationis et Partitionis Compendiis Miscellaneis," after having published the Logarithms, without the slightest allusion to that means of substituting, for the real numbers, a set of artificial numbers rendering the whole art of calculation compendious beyond the most sanguine hopes of emancipated Astronomy.

It is even more improbable that the treatises in question were composed, or so far reduced to order, at some period in the long interval which must have elapsed between the author's conception of the Logarithms, and the construction of those elaborate works by which the glory of the invention was secured to himself, and the benefit bestowed upon mankind. In his book of Geometrical Logistic, Napier seems to allude to the Arithmetic of Surds as a discovery of his own, the full value of which he promises afterwards to disclose. Had he been conscious at this time of the far more important secret of the Logarithms,—which, after his first conception of the idea, was his daily labour to the end of his days,—there would, in all probability, have been some allusion to it. Indeed, his great invention must have formed a most interesting chapter of his work De Arte Logistica, had he lived to complete it.

Not only is there no mention of his Canon of the Logarithms in these treatises, but when, to illustrate the relation between the powers of numbers and their indices, there is exhibited the following table,—

	I.	II.	III.	IIII.	V.	VI.	VII.
1.	2.	4.	8.	16.	32.	64.	128,—

no allusion occurs to the fact, that this table also affords an illustration of the logarithmic principle. The upper series are not only indices to the lower series, considered as powers, but they are Logarithms to those numbers, being an arithmetical adapted to a geometrical progression. The word Logarithms was compounded by Napier before he published his discovery, and this term itself embodies the principle illustrated (though on a very limited scale) by the above arrangement of figures. Αριθμοι signifies numbers; λογαριθμοι, the ratios of numbers, or rather the number of ratios, λογων αριθμος. Now, although Napier expounded his system of Logarithms through his geometrical idea of fluxions (a demonstration and a term afterwards adopted by Newton), and not through the doctrine of powers and exponents, yet he could not have composed the term Logarithms without a perfect knowledge of the connexion of his system with the arithmetic of proportions.

It can scarcely be doubted, then, that the composition of Napier's treatise of Numbers is prior in date even to his conception of the Logarithms; a conclusion that will seem very natural to any one who has examined his works on the subject, and who is at all capable of appreciating the command of the science of Numbers which those works evince. Undoubtedly, his great invention was not (as is alleged of the theory of gravitation) the result of a sudden thought or suggestion, in some happy moment of genius. It was laboriously elicited from the arcana of Numbers,—deliberately extracted, by a sort of Cæsarean operation, from the unripe womb of analytical science. Napier himself alludes to this predetermination of his genius, in a most interesting letter, the last he ever wrote. It is prefixed as a dedication, to the Chancellor of Scotland, of his Rhabdologia, or the art of computing by means of figured rods, better known by the name of "Neper's bones." In the present attempt to trace the chronological order of his works, I may here take the liberty to translate this letter from the original Latin, the language in which he composed all his mathematical works:—

"To the most illustrious Alexander Seton, Earl of Dunfermline, Lord of Fyvy and Urquhart, High Chancellor of Scotland, &c.

"The difficulty and prolixity of calculation (most illustrious Sir), a toil which is apt to deter most people from the study of mathematics, I have, all my life, with what powers and little genius I possess, laboured to eradicate. And, with that end in view, I published of late years the Canon of Logarithms (for a long period elaborated by me), which, rejecting the natural numbers, and the more difficult operations performed by them, substitutes in their place others affording the same results by means of easy additions, subtractions, and extractions of roots. Of which Logarithms, indeed, I have now found out another species much superior to the former, and intend, if God shall grant me longer life, and the possession of health, to make known the method of constructing, as well as the manner of using them. But the actual computation of this new Canon, on account of the infirmity of my bodily health, I have left to some who are well versant in such studies; and especially to that most learned man, Henry Briggs, public professor of geometry in London, my most beloved friend. Meanwhile, however, for the sake of those who may prefer to work with the natural numbers as they stand, I have excogitated three other compendious modes of calculation. The first is by means of numerating

rods, and this I have called RHABDOLOGIA. Another, by far the most expeditious of all for multiplication, and which, on that account, I have not inaptly termed the PROMPTUARY OF MULTIPLICATION, is by means of little plates of metal disposed in a box. And, lastly, a third method, namely, LOCAL ARITHMETIC performed upon a chess-board. But, to the publication of this little work, concerning the mechanism and use of the rods, I was specially impelled, not merely by the fact that they are so approved of as to be already almost in common use, and even carried to foreign countries; but because it also reached my ears that your kindness advised me so to do, lest they should be published in the name of another, and I be compelled to sing with Virgil,

'Hos ego versiculos feci,' &c.

This very friendly counsel from your Lordship ought to have the greatest weight with me; and most assuredly, but for that, this little treatise of the use of the rods (to which the other two compendious methods are added, by way of appendix), would never have seen the light. If, therefore, any thanks be due from the students of mathematics for these little books, they all belong to you as your just right, my noble Lord; to you, indeed, they must spontaneously fly, not only as patron, but a second parent: especially since I am assured that you have done these

rods of mine such high honour as to have them framed, not of common materials, but of silver. Accept, then, my Lord, in good part, this small work, such as it is ; and, though it be not worthy of so great a Mæcenas, take it under your patronage as a child of your own. And so I earnestly pray God to preserve you long, to us and the state, to preside over justice and equity.

"Your Lordship's very much obliged

"JOHN NAPIER,

"Baron of Merchiston."

The date of the volume to which this letter is prefixed is 1617, and Napier died upon the 4th of April in that same year. We learn from it the chronological order of the composition of all his published works. In the first place, he had long and laboriously wrought out—"à me longo tempore elaboratum"—his Canon of the Logarithms. Then he excogitated—"excogitavimus"—the mechanical system of calculating rods, for the sake of those who might be distrustful of his artificial system of Logarithms. His Promptuary of Multiplication, which immediately follows the Rhabdologia, he states to be the latest of all his inventions—"omnium ultimo à nobis inventum sit hoc Multiplicationis Promptuarium"—though he gives it precedence in the volume, according to his estimate of its importance. He had previously invented his mode of calcu-

lating with the abacus or chess-board, in the preface of which he again refers to the origin of all these inventions,—namely, that it was the constant labour of his life to rend the fetters with which science was yet subdued. " In the course," says he, " of devoting every moment of my leisure,"—doubtless, from what he considered his great calling, the exposition of the Revelations,—" to the invention of these compendious methods of calculation, and to the enquiry by what means the labour and toil of calculation might be removed, besides the Logarithms, Rhabdologia, the Promptuary of Multiplication, and other devices, I hit upon a certain arithmetical table, which, as it performs the more troublesome operations of common arithmetic upon an abacus or chess-board, may be considered an amusement rather than labour; for, by means of it, addition, subtraction, multiplication, division, and even the extraction of roots, are accomplished, simply by moving counters hither and thither upon the board. Unwilling either to consign it to oblivion, or to publish so small a matter by itself, I have prefixed it to my Rhabdologia, in addition to the Promptuary, for the benefit of the studious—and the criticism of the learned."

That Napier composed his Rhabdologia after the year 1614, when he published his Canon of the Logarithms, is also indi-

cated by the fact, that in the minor work (Lib. I. cap. 2) he selects, as a numerical example, "Annus Domini 1615." Now, supposing the Rhabdologia an invention of earlier date than the Logarithms (which latter, as will presently appear, must be referred to some time in the previous century), the year of our Lord 1615, when thus selected for the example, was a year yet to come; and it is not likely that such an example would have occurred to him, in place of a year either present or past. Probably the year selected was that in which Napier was writing at the time, a supposition perfectly consistent with his statement, in the letter to the Chancellor, that this minor invention was for the benefit of those who might be disinclined to use the Logarithms, and that it had become well known, and even been carried abroad, before he was induced to publish it. For 1615 was the year after the publication of the Logarithms, and the Rhabdologia was not printed until 1617.

There is only one circumstance which seems to interfere with this chronology; but it is easily explained. In the fifth chapter of the second book of his Arithmetic, when expounding various compendious methods of multiplication and division, Napier adds, "Sive omnium facillime per ossa Rhabdologiæ nostræ." This might be supposed to afford decisive evidence that the Rhabdologia was composed before the book of Arith-

metic. It must be observed, however, that the expressions quoted are written on the margin of Robert Napier's manuscript, and there is a mark in the text where the note should be inserted,—a direction complied with in printing the present volume (p. 42). Probably this note was an addition of the transcriber's ; or perhaps Napier himself had subsequently added it to his MS., after the invention of the calculating rods had occurred. If the note be contemporary with the composition of the treatise, then undoubtedly the latter is of a subsequent date to his Rhabdologia. But there is another circumstance which seems to render this supposition totally out of the question. In Napier's Arithmetic occurs a genesis of decimal fractions, and perhaps the earliest on record. But the peculiar notation of decimals,—from which the system derives all its power as a reciprocation of the Arabic scale, and which Napier could not have failed to use and comment upon in any notice of a decimal fraction, when once aware of the expedient,—is not hinted at in the treatise on Arithmetic. Now, in the Rhabdologia, there is a section (of the fourth chapter) entitled " Admonitio pro Decimali Arithmeticâ,"—being his exposition of that very notation of decimals which is in use at the present day. Napier had hit upon this admirable expedient when constructing his Canon of the Logarithms,—the first mathematical work in which the arithmetic of decimal fractions was

developed, and displayed in full operation. This circumstance of itself appears to determine that the composition of his treatise on Arithmetic is prior in date both to the invention of the Logarithms and of the Rhabdologia.

It is very probable that Napier's progress, in composing a complete digest of Numbers, might have been interrupted, and that work set aside for the time, in consequence of those very operations having suggested his more profound inventions; but that he composed his Arithmetic and Algebra subsequently to those inventions is most improbable, considering that the dates of his published works are 1614 and 1617, and that he died in the month of April of the latter year. Moreover, it appears that, at the time of his death, Napier was occupied in bringing to perfection his most elaborate and beautiful work, the "Constructio Logarithmorum;" which treatise, although it did not see the light until two years after his death, when it was edited by his son Robert, he had left in a state for publication. This volume, with some profound and original aids to the science of trigonometry added as a supplement to it, must have occupied all the time which he devoted to mathematics, (for he was still deeply engaged with the Apocalypse,) between the date of the publication of the Canon of Logarithms and the date of his death. The

concluding sentences of his Canon afford so interesting a view of the last labours of Napier's life, that I may here quote them from the English translation of 1616, which he revised himself:—

"Now, therefore, it hath been sufficiently showed that there are Logarithms, what they are, and of what use they are; for with help of them, we have both demonstratively showed, and taught by examples of both kinds of trigonometry, that the arithmetical solution of any geometrical question may most readily be performed without trouble of multiplication, division, or extraction of roots. You have, therefore, the admirable table of Logarithms that was promised, together with the most plentiful use thereof, which, if (to you of the learned sort) I by your letters understand to be acceptable to you, I shall be encouraged to set forth also the way to make the table. In the mean time, make use of this short treatise, and give all praise and glory to God, the high Inventor and Guider of all good works."

Then follows an isolated sentence which he terms "Admonitio;" and, probably, he refers to his theological labours when, in that sentence, he speaks of "rerum graviorum cura" having interfered with the perfecting of his mathematical work:—

"Seeing that the calculation of this table, which ought to have been perfected by the labour and pains of many calculators, has been accomplished by that of one alone, it will not be surprising if many errors have crept into it. I beseech you, benevolent readers, pardon these, whether occasioned by the weariness of calculation, or an oversight of the press; for as for myself, the infirm state of my health, and weightier occupations, have prevented my adding the last finish. But if I shall understand that the use of this invention proves acceptable to the learned, I may, perhaps, in a short time (God willing) publish the philosophy of it, and the method either of amending this Canon, or of constructing a new one upon a more convenient plan; that thus, through the diligence of many calculators, a Canon, more highly finished and accurate than the labour of a single individual could accomplish, may at length see the light. Nothing is perfect at its birth."

Yet the work had been long and intensely laboured, and came from his hands nearly perfect. No sooner had KEPLER perused it, than he addressed his enthusiastic letter to this "most illustrious Baron," as he calls him,—the mysterious hermit of the sciences, whose startling appeal to the great continental astronomers, issuing from the terra incognita of

Scotland, had come upon the immortal German like a voice from another world. Ere Kepler's letter reached Scotland, Napier was no more. The response came not from "The Baron of Merchiston," who was no longer to be found in his old tower on "the Borough-muir," nor in his yet more romantic retreats in the Lennox and Menteith. Had he been spared to reply, and to express his delight at the homage of Kepler, it might have been said, of this eloquent and electric sympathy between the distant orbits of Kepler and Napier,—

> "Rude Scotia's mountains now have found a tongue;
> Benlomond answers, through his misty shroud,
> Back to the joyous Alps, who call to him aloud."

But, to return to a less poetical view of the matter, Robert Napier, the transcriber of the manuscripts now published, was he who responded to Kepler's ardent request, by publishing his father's "Logarithmorum Canonis Constructio;" and, to complete this review of the nature and chronological order of Napier's mathematical studies, I shall here translate the preface to his posthumous work, which was published in 1619:—

"Some years ago, my father, of ever venerated memory, published the use of the Wonderful Canon of Logarithms; but the construction and method of generating it, he, for certain

reasons, was unwilling to commit to types, as he mentions upon the seventh and the last pages of the Logarithms, until he knew how it was judged of and criticised by those who are versed in this department of letters. But since his death I have been assured, from undoubted authority, that this new invention is much thought of by the most able mathematicians, and that nothing would delight them more than if the construction of his Wonderful Canon, or so much at least as might suffice to illustrate it, were published for the benefit of the world. Although, therefore, it is very manifest to me that the author had not put his last finish to this little work, I have done what in me lay to satisfy their laudable desires, as well as to afford some assistance, especially to those who are weak in such studies, and apt to stick at the very threshold. I doubt not, however, that this posthumous work would have seen the light in a far more perfect and finished state, if to the author himself, my dearest father,—who, according to the opinion of the best judges, possessed among other illustrious gifts this in particular, that he could explicate the most difficult matter by some sure and easy method, and in the fewest words,—God had granted a longer use of life. You have, then (benevolent reader), the doctrine of the construction of Logarithms, which here he calls artificial numbers,—for he had this treatise beside him composed for several years before he invented the word

Logarithms,—most copiously unfolded, and their nature, accidences, and various adaptations to their natural numbers, perspicuously demonstrated. I have also thought good to subjoin to the Construction itself a certain appendix, concerning the method of forming another and more excellent species of Logarithms, to which the inventor himself alludes in his epistle prefixed to the Rhabdologia, and in which the Logarithm of unity is 0. The treatise which comes last is that which, tending to the utmost perfection of his Logarithmic trigonometry, was the fruit of his latest toil,—namely, certain very remarkable propositions for resolving spherical triangles, without the necessity of dividing them into quadrantal or rectangular triangles, and which are absolutely general. These, indeed, he intended to have reduced to order, and to have successively demonstrated, had not death snatched him from us too soon. I have added some lucubrations upon these propositions, and also upon the new species of Logarithms, calculated by that most excellent mathematician, Henry Briggs, public professor in London, who undertook most willingly the very severe labour of this Canon, in consequence of the singular affection that existed between him and my father of illustrious memory,—the method of construction and explanation of its use being left to the inventor himself. But now, since he has been called from this life, the whole burden of the business rests upon the learned

Briggs, as if it were his peculiar destiny to adorn this Sparta. In the mean while, reader, enjoy these labours such as they are, and receive them in good part. Farewell.

"ROBERT NEPER."

From this history of Napier's studies, derived chiefly from himself and his son, the conclusion is inevitable, that the work now given to the public was composed before his conception of the Logarithms; and that that invention, as well as his subsequent minor inventions, arose out of the very attempt, to abridge the toils of calculation, which he indicates in the fifth chapter of the second book of his Arithmetic, " De Multiplicationis et Partitionis Compendiis Miscellaneis." In order to ascertain, then, the age of his Arithmetic and Algebra, we must determine, as nearly as possible, the period of time when he formed his conception of the Logarithms.

No one who has taken the trouble to examine Napier's original works, or who understands any thing of the nature and construction of Logarithms, can doubt that many years of labour must have elapsed, between the first idea of the system in the inventor's mind, and its publication, in 1614, accompanied with calculated tables of Logarithms. The work published after his death had been composed for several years, as

we are informed by his son, ere Napier had compounded the term Logarithms, under which denomination, however, they were first published. He himself alludes to the long labour bestowed upon his Canon, and to the interesting fact that his operose calculations were entirely performed by himself. M. Biot, in his review and " extrait " of the Memoirs of Napier, published in the Journal des Savans for the year 1835, after passing the highest encomiums upon the genius of Napier's works, thus speaks of the mechanical labour of the calculations :—

" Besides the merit of the invention, Napier's tables are a prodigy of laborious patience. As we reflect upon the time and toil it must have cost him to calculate all those numbers, we shudder at the chances there were of his being cut off ere he had realized his idea, and of its having died with him. It has been said, and Delambre repeats the observation, that the last figures of his numbers are inaccurate. This is true ; but it would have been a more valuable truth to have ascertained whether the inaccuracy resulted from his method, or from some error of calculation in applying it. This is what I have done ; and I have discovered an error of the kind, a very slight error, in the last term of the second progression, which he forms preparatory to the calculation of his tables. All the successive

steps being deduced from this one, the trifling error in question is thus carried on through the calculations. I corrected his error, and then, adopting his own method, but abridging the operations by our more rapid processes of developement, I calculated the logarithm of 5000000, which is the last in Napier's table, and consequently that upon which all the errors accumulate. I found for its value 6931471.808942; whereas, by the modern series, it ought to be 6931471.805599. Thus the difference commences at the tenth figure. I calculated, in like manner, the hyperbolic logarithm, with Napier's numbers corrected, and found for its value 2.3025850940346, while, by our present tables, it is 2.3025850929940; so the real difference only falls on the ninth decimal. But this is beyond the range of the tables of Callet, which are in daily use. If Napier had even commanded the services of a village schoolmaster, to calculate, by the inventor's own method of subtractions, a geometrical progression still slower than what he used, a desideratum to which he himself calls attention, the tables of Briggs, calculated to fourteen decimals, would have possessed no superiority over those of Napier. His minor inventions are scarcely worth mentioning after this immense invention of Logarithms. That was sufficient for the lifetime, as it is for the fame, of a single individual."

It is interesting to compare this judgment upon Napier's labours, pronounced in the nineteenth century from the highest tribunal of science, with that modest "Admonitio" of his own, already quoted, wherein he anticipates the chance, and deprecates the criticism, of casual errors in his unaided calculations. Biot tells us, that he could only discover "une petite faute de ce genre, une très petite faute." We may assume, then, that Napier's great invention had first occurred to him many years before its publication. Indeed the following details, which will serve to determine the chronology both of the invention of Logarithms and of the treatises now submitted to the public, afford a curious proof that he was in possession of the secret, and preparing his publication of the Logarithms, some time before the close of the sixteenth century.

It is a remarkable fact, not generally known, that Tycho Brahe obtained some hint of this boon to be conferred upon science, twenty years before Napier's other avocations, added to the tedious labour of the calculations and his own diffidence, permitted him to give his invention to the world. Sir Archibald Napier of Merchiston, the father of the great Napier, was Justice-Depute in the reign of Mary of Scotland. His colleague in that office was Sir Thomas Craig of Riccarton, celebrated for his work on Feudal law.

The third son of the Feudist was John Craig, afterwards Dr John Craig, physician to James VI.; and he was also the friend of John Napier, the son of his father's colleague in office. But there was another, and perhaps the stronger tie. Young Craig was much given to mathematical studies. There is one record, indeed, of his success in those pursuits, which of itself is sufficient to distinguish his memory in that respect,—a record rarely met with in his own country, and still seldomer perused. I allude to a small volume of Latin epistles, printed at Brunswick in the year 1737, and dedicated by their collector, "Rud. Aug. Noltenius," to the Duke of Brunswick. The three first letters in this collection are from Dr John Craig to Tycho Brahe, and they prove that he was upon the most friendly and confidential footing with the Danish astronomer. He addresses Tycho as his "honoured friend," and signs himself, "your most affectionate John Craig, Doctor of Philosophy and Medicine." The first of these letters (which are written in Latin) commences with the fact, that, "About the beginning of last winter that illustrious man, Sir William Stewart, delivered to me your letter, and the book which you sent me." The date of this letter is not given; but, in the Library of the University of Edinburgh, there is a mathematical work by Tycho Brahe, which bears, upon the first blank leaf, a manuscript

sentence in Latin to the following effect: "To Doctor John Craig of Edinburgh, in Scotland, a most illustrious man, highly gifted with various and excellent learning, Professor of Medicine, and exceedingly skilled in the Mathematics, Tycho Brahe hath sent this gift, and with his own hand hath written this at Uraniburg, 2d November 1588."

It appears from contemporary records, that, in the month of August 1588, Sir William Stewart, commanding the royal guard of Scotland, was sent to Denmark to arrange the preliminaries of King James's marriage, and that he returned to Edinburgh upon the 15th of November 1588. There can be no doubt that the volume above-mentioned is that referred to in Craig's epistle to Tycho. Neither can it be doubted that this was Dr John Craig, third son of Sir Thomas Craig of Riccarton, and physician to James VI. He was raised to his highest post in the royal household by a gift from that monarch, recorded in the Fœdera, "dilecto nobis Johanni Craigio in medicinis doctori, officium et locum ordinarii, et primarii medici nostri," dated at Westminster, 20th June, 1603. This was the friend of Tycho Brahe and of Napier. The Danish astronomer transmitted his present to John Craig, from his palace of Uraniburg, with the air of the Monarch of Science. Napier, from his old tower of Merchiston, when writing, in

the year 1608, to his own son, who was gentleman of the bed-chamber to King James in England, thus remembers the King's physician:—" Ye sall make my commendatiouns to Doctor Craig."

The facts now stated serve to throw light upon a sentence, respecting the invention of Logarithms, which occurs in one of Kepler's letters,—a sentence which has been little noticed, and never rightly understood. Kepler, when writing to his favourite correspondent Petrus Cugerus (a mathematician of Dantzic, and the master of Helvelius) upon the subject of the economy of the heavenly bodies, and after reveling in his deepest calculations; declares that, of all the methods of calculation in aid of astronomy, nothing excels the invention of Napier; and yet (he adds), even so early as the year 1594, a certain Scotchman, in a letter written to Tycho, held out some promise of that Wonderful Canon. The precise words of Kepler's letter are, " Nihil autem supra Neperianam rationem esse puto: etsi, quidem, Scotus quidam, literis ad Tychonem A. CIϽIϽXCIV. scriptis, jam spem fecit Canonis illius Mirifici." Kepler was the pupil of Tycho, and joined him as such after the reverse of fortune which expelled the illustrious Dane from his palace of Uraniburg. This accounts for Kepler's having obtained some partial knowledge of the letter to which he alludes. But who

was the "certain Scotchman" from whom Tycho received this important communication before the year 1594,—a date so precisely given by Kepler, that we cannot doubt its accuracy?

John Craig had long intended to pay a visit to Tycho Brahe. This appears by his letter of the year 1589, already quoted, in which he states that five years before he had made an attempt to reach Uraniburg, but had been baffled by storms and the inhospitable rocks of Norway; and that ever since, being more and more attracted by the accounts, brought by ambassadors and others, of Tycho's fame, and the magnificence of his observatories, he had been longing to visit him. In the year 1590, James VI., the patron of Craig, spent some days at Uraniburg, before returning to Scotland from his matrimonial expedition. It cannot be doubted that James's physician, who was long about this monarch's person in a medical capacity, and eventually at the head of his medical staff, would seize the propitious opportunity, of the progress of his royal master, to visit his friend; and it is not unlikely that Craig himself had suggested or encouraged the visit of his royal master to Uraniburg. Dr Craig, it is most probable, was the "certain Scotchman" to whom Kepler alludes, and who, in the year 1594, had written a promise of the

Logarithms to Tycho Brahe. For after his visit to Tycho, in the year 1590, the person in Scotland, to whom Craig would most eagerly unfold the wonders of Uraniburg, was Napier of Merchiston.

Frederick II. of Denmark, the munificent patron of Tycho, had established that philosopher on the island of Huen, situated at the mouth of the Baltic, adding honours and revenues, and every aid and encouragement which the most ardent astronomer could desire. Upon the 8th of August, 1576, the first stone of the far-famed castle or palace of Uraniburg was laid in Tycho's principality. It was a vast quadrangle, the dimensions being sixty feet every way, and flanked with lofty towers thirty-two feet in diameter, the observatories of this palace of science. Tycho is also said to have fitted it up with certain mysterious tubes, and other telegraphic contrivances, which enabled him to communicate with his domestics as if by magic, and obtain secret knowledge of his many visitors long before their arrival.

In the year 1590, that in which King James visited Tycho, Napier's fertile genius, unaided by the encouragement of royal patronage, was teeming with various discoveries in mechanical science (besides his speculations in the science of Numbers),

which, like Archimedes of old, he intended should be applied for the defence of the island against foreign invasion. A report made to Napier by his friend Dr Craig, of all he had seen and heard at Uraniburg, when there with his Majesty, would perfectly account for the following sentence of Napier's admonitory letter to King James (on the subject of his Majesty's supposed inclination to Popery), dated from Merchiston, 29th January, 1593:—" For let not your Majesty doubt but that there are within your realm, as well as in other countries, godly and good ingynes, versed and exercised in all manner of honest science, and godly discipline, who, by your Majesty's instigation, might yield forth works and fruits worthy of memory, which otherwise, lacking some mighty Mæcenas to incourage them, may perchance be buried with eternal silence."

His conversations with Dr Craig might suggest to Napier, not merely this hint to King James, that his Majesty should patronise science in Scotland, but the transmission of a hint to Tycho himself on the subject of astronomical calculation. If the King of Denmark, when he invested Tycho with something like eastern splendour, could have added the power that still lay hid in Arabic numbers, a false system of the world would not have been re-established at Huen. It was in the

means of astronomical calculation that the science of Tycho Brahe was most deficient. Now Kepler tells us, that even so early as in the year 1594, (the year following that in which Napier's letter to King James is dated,) a certain Scotchman had written to Tycho some promise of the Logarithms. After the facts referred to, it can scarcely be doubted that this correspondent from Scotland was none other than Tycho's old friend and correspondent there, the learned physician of King James, and also the friend of Napier, with whom he could not fail to have often discussed the subject of the royal visit to Uraniburg. And this appears to be placed beyond all doubt, when, to what has been already stated, we add the following anecdote, somewhat imperfectly recorded by Anthony-à-Wood.

That indefatigable and amusing collector of literary gossip thus narrates it, in the Athenæ Oxonienses :—" It must now be known that one Dr Craig, a Scotchman, coming out of Denmark into his own country, called upon Joh. Neper, Baron of Marcheston, near Edinburgh, and told him, among other discourses, of a new invention in Denmark, by Longomontanus, as 'tis said, to save the tedious multiplication and division in astronomical calculations. Neper being solicitous to know further of him concerning this matter, he could give no

other account of it than that it was by proportional numbers. Which hint Neper taking, he desired him at his return to call upon him again. Craig, after some weeks had passed, did so, and Neper then shewed him a rude draught of what he called Canon Mirabilis Logarithmorum. Which draught, with some alterations, he printing in 1614, it came forthwith into the hands of our author Briggs, and into those of Will. Oughtred, from whom the relation of this matter came."

Any one at all conversant with the history of science, and the nature of the invention of Logarithms, will at once perceive, that, whatever foundation in fact this anecdote may have, it is here inaccurately and ignorantly told. Longomontanus was the pupil and assistant of Tycho at Uraniburg, and a most distinguished mathematician. He, in common with many others, was well acquainted with a principle of numerical progression, the extraordinary generalization of which by Napier is that which constitutes the invention of Logarithms. Archimedes had first observed and speculated upon such progressions, but without discovering the Logarithms. It could not possibly be, that a hint from Longomontanus had suggested to Napier his great invention; for if a hint of the kind could have urged any human intellect thus rapidly upon the conception of the Logarithms themselves, that hint had arisen in the

school of Alexandria, was submerged in the middle ages, and rose again with the letters of Greece,—a hint which Tycho had, which Stifellius, Byrgius, Longomontanus, and Kepler himself had; yet no more was made of it after the revival of letters, than had been by Archimedes before their fall. Kepler too, informs us, that in the year 1594, something regarding the generalization of this numerical principle had actually been reported in a letter to Tycho; yet the secret was still undiscovered until Napier published his Canon in 1614. In a letter, dated 11th March, 1618, to his friend Schikhart, Kepler, after descanting upon the various difficulties and resources of trigonometry, exclaims,—" A Scotish Baron has started up, his name I cannot remember, but he has put forth some wonderful mode by which all necessity of multiplications and divisions is commuted to mere additions and subtractions, nor does he make any use of a table of sines; still, however, he requires a canon of tangents, and the variety, frequency, and difficulty of additions and subtractions, in some cases exceed the labour of multiplication and division." This was the first crude and inaccurate idea formed by Kepler of the work which he had not yet studied; and already the Scotish Baron whose name he could not remember was in his grave! But of this fact Kepler was not aware even on the 28th July, 1619, when he thus addressed Napier himself:—

"But the chief cause that impeded my progress this year, in framing the Rudolphine Tables, was an entirely new, but happy calamity, which has befallen a part of the Tables I had long ago completed. I mean, most illustrious Baron, that book of thine, which, published at Edinburgh in Scotland five years ago, I first saw at Prague the year before last. At that time I had not leisure to study it; but last year, having met with a little book by Benjamin Ursin (long my familiar, and now astronomer to the Margrave), wherein he briefly gives the marrow of the matter, extracted from your own work, I was awakened to its merits. Scarcely, indeed, had I made trial of it, in a single example, when I became aware, to my great delight, that you had generalized a certain play of Numbers, which I myself, in a very minute degree, had practically adopted for many years, and had proposed to incorporate with my Tables; especially in the matter of parallaxes, and in the minutes of duration and delay in eclipses; of which method this very Ephemeris exhibits an example. I knew, indeed, that this method of mine was only applicable in the single case of an arc differing in no sensible degree from a straight line. But of this I was ignorant, that, from the excesses of the secants, LOGARITHMS could be generated, thus rendering the method universal throughout any extent of arc. Then indeed my mind, above all things, was eager to ascertain, whether,

in this little book of Ursin's, these Logarithms had been accurately investigated. Calling to my aid, therefore, Janus Gringalletus Sabaudus, my familiar, I ordered him to subtract the thousandth part of the whole sine,—again, to subtract the thousandth part of that residue, and to repeat this operation more than two thousand times, until there remained about the tenth part of the whole sine; but of the sine from which a thousandth part had been subtracted, I computed the logarithm with the greatest care, beginning from the unit of that division which Pitiscus most frequently uses, namely, the duodecimal. The logarithm thus computed, I arranged uniformly with the remainders of all the subtractions. In this manner, I ascertained that these logarithms were as nearly perfect as possible, although a few errors had crept in, either of the press, or in that minute distribution of the greater logarithms about the beginning of the quadrant. I mention all this to you by the way, in order that you may understand how gratifying it would be, to me at least (and I should think to others), if you would put the world in possession of the methods by which you proceeded. Of these, I make no doubt, you have many, and most ingenious, at your command intuitively. And so the promise held out by you on the 57th page of your work," (to put the world in possession of his methods of constructing the Logarithms, should he understand that the invention was

approved of by men of science,) "has fallen due to the public. And now let us grapple with your Tables."

This interesting letter, written after he to whom it was addressed had been dead for nearly two years, affords the most complete refutation of that part of the anecdote, in the Athenæ Oxonienses, which seems to impute Napier's invention to a sudden thought suggested by a hint from Kepler's intimate friend and companion Longomontanus. Kepler himself, in this letter, refers the invention to its true source, namely, Napier's surpassing command of the science of Numbers. Kepler, whose own immortal discoveries in Astronomy obtained for him the daring title of Legislator of the Heavens, bowed at the shrine of the Logarithms, and at once, and without reserve, acknowledged Napier as his master in Logistic. A quaint indication, of his estimate of Napier's invention, is also to be discovered in the frontispiece to those Rudolphine Tables of which he speaks in his letter. The telescope of Galileo, the elliptical orbit of a planet, the system of Copernicus, and a female figure with the Napierian Logarithm of half the radius of a circle arranged as a glory round her head, are there delineated as figurative of the mighty impulses which Astronomy had received in those days. But that the anecdote of Anthony-à-Wood has some foundation in fact is obvious, when

we compare it with what has been previously stated. The Doctor Craig therein mentioned is undoubtedly Napier's friend, the physician of King James; and the "coming out of Denmark into his own country," refers to his return with his Majesty in 1590. Moreover, that Craig had some conversation with his friend the Baron of Merchiston on the subject of astronomical calculations, and that the result was some communication on the subject of the Logarithms, seems to be verified by the independent evidence afforded by Kepler's other letter, where he says that a certain Scotchman, in a letter to Tycho written in the year 1594, already held out some hopes of that wonderful Canon; a fact which Kepler had subsequently become aware of, probably from his inspection of Tycho's scientific papers and correspondence. Thus, by a very curious chain of evidence, it seems proved, that Napier had made some advance in his construction of the Logarithms so early as 1594, and that he had set himself deliberately to the task, after the difficulties which harassed the throne of Astronomy at Uraniburg (in consequence of the crudeness of numerical science) had been reported to him in 1590. But, as Kepler himself observes, the Inventor of Logarithms must have previously commanded many and most ingenious numerical resources. He must have been far in advance of his age in a profound knowledge of the mysterious play of Numbers. His

mind must have been thoroughly and deeply imbued with the Logistic art, and analytical power, ere he proposed to himself the invention of that great lever of science. Now it is in the Latin treatises, which these observations preface, that we may trace Napier's command of Numbers, before he had formed a conception of the Logarithms. And thus the present work may be said to afford a most interesting chapter in the history of the progress of Science. A digest of the whole art of Logistic, equal, in so far as the then existing numerical notation admits of the comparison, to Euler's in modern times, composed prior to the year 1594, amid the storms of faction which then distracted Scotland, was an achievement worthy of being crowned by the invention of Logarithms, for the natural discovery of which, in the gradual development of algebraic resources, the analytic art was at the time an age too young.

Another circumstance may here be mentioned which also affords evidence, that, many years prior to the publication of the Logarithms, Napier's fame as a mathematician was established with the learned of his own countrymen, and that he was the person to whom Dr Craig would be most eager to converse on the subject of his visit to Tycho Brahe. The celebrated lawyer, and learned and accomplished author, Sir

John Skene of Currie Hill, when, in the course of preparing his treatise " De Verborum Significatione" (first published in the year 1597), he came to the word " particata vel perticata terræ," which he explains, " from the French word ' perche,' meikle used in the English lawes, ane ruid of land," adds this sentence :—" But it is necessare that the measurers of land called landimers, in Latin ' agrimensores,' observe and keep ane just relation betwixt the length and the breadth of the measures quhilk they use in measuring of lands, quhairanent I find na mention in the lawes and register of this realme, albeit ane ordinance thereanent be made by King Edward the First, King of England, the 33d yeir of his reigne ; and because the knawledge of this matter is very necessare in measuring of lands dayly used in this realme, I thought good to propone certaine questions to John Naper, fear of Merchistoun,—ane gentleman of singular judgement and learning, especially in the mathematical sciences,—the tenour quhairof, and his answers made thereto, followis." Napier's answers are of considerable length, and given with his characteristic simplicity and power.

If the composition of the treatises now published is to be referred to a date prior to the year 1594, which seems to be placed beyond doubt by what has been stated, they are

interesting not merely as the works of the Inventor of Logarithms, but as being the very first of the kind in Scotland, and among the first systematic works on Numbers after the revival of letters in Europe. The sixteenth century was the rudest period of algebraic science in Europe. Leonardo of Pisa, indeed, composed his work before the invention of printing, and early in the thirteenth century; but this had been lost sight of, and was not known for more than a century after Napier's death, when the manuscript was discovered at Florence. The first printed work on the subject was that of Lucas de Burgo, from whom is generally dated the decided dawn of Algebra in Europe. De Burgo's principal work was printed about the year 1494. The second printed work upon Arithmetic and Algebra appeared in 1539. This was a work of the great but eccentric Cardan, of whom it is affirmed by Scaliger that he was so devoted to astrology as actually to starve himself to death, that his own astrological prediction might be fulfilled,—a very equivocal illustration of his favourite science. Cardan died in the year 1575. Germany produced one or two mathematicians, who, at the same time that Cardan wrote, gave a more decided impulse to Numbers. Hitherto nothing had been added to that recondite science, since the introduction of the Arabic numerals, and the rude and imperfect symbols of Burgo's Algebra, except in the

theory of equations, which had received a great extension from Tartalea and Cardan. The defect which materially impeded the system was in notation, the mainspring of numerical science. In the year 1544, Michael Stifellius, a Lutheran clergyman, published at Nuremberg his Arithmetica Integra, a very original Latin treatise on Arithmetic and Algebra, wherein he viewed numerical quantities, and their combinations, closely and ingeniously, and gave an impulse to Algebra by improving its notation. He was the first to introduce the signs + and — for plus and minus, and also the character $\surd$ (derived from the letter R), to denote the radix or root. Moreover, he entered systematically into the consideration of arithmetical and geometrical progressions, pointing out what may now be termed the logarithmic properties of a corresponding series of the powers of a given number, and the exponents of those powers, which latter term he uses. But in this speculation he had formed no conception of the possibility of changing the infinite series of natural numbers from an arithmetical to a geometrical progression, and then of generating a corresponding arithmetical progression.

In the year 1552 appeared the first treatise upon Arithmetic and Algebra in the English language. Its author was the unfortunate Robert Recorde, a mathematician of un-

doubted genius, but so little appreciated that he was allowed to linger out his last days in the Fleet, where he was imprisoned for debt, and where he died about the year 1558. His works are curious and original, but rude and puerile in their style. They are generally in the form of a dialogue between a Master and Scholar, and under such quaint titles, as " The Pathway to Knowledge "— " The Ground of Arts "—" The Castle of Knowledge "—" The Whetstone of Wit." Napier's style of communicating even his elementary rules is clear, simple, and philosophical. His son tells us, that, " according to the opinion of the best judges, my dearest father possessed, amongst other great endowments, this in particular, that he could explicate the most difficult matter by some sure and easy method, and in the fewest words." Accordingly, he never indulged in such facetiæ as the following, which occurs in Recorde's Arithmetic, a work, nevertheless, of original genius :—" Master. ' Exclude number, and answer this question—How many years old are you ?'—Scholar. ' Mum.'—Master. ' So that, if number want, you answer all by mummes. How many miles to London ?'—Scholar. ' A poak full of plums.'—Master. ' If number be lacking, it maketh men dumb ; so that, to most questions, they must answer mum. What call you the science you desire so greatly ?'—Scholar. ' Some call it Arsemetrick, and some Augrime.'—

Master. 'Both names are corruptly written; Arsemetrick for Arithmetic, as the Greeks call it; and Augrime for Algorisme, as the Arabians sound it.'"

Recorde was ingenious and inventive, and is remarkable, in the history of Algebra (which owes so much to successive inventions in the art of notation), for having added to its characters the sign of equality. Mr Babbage had overlooked the fact. That distinguished mathematician, in his history of notation, observes, "It is a curious circumstance that the symbol which now represents equality was first used to denote subtraction, in which sense it was employed by Albert Girard, and that a word signifying equality was always used instead, until the time of Harriot." The works of Girard and Harriot did not appear until the seventeenth century was far advanced, and long after Napier's death. Napier's book of Arithmetic, as already shown, must be referred to a date prior to the year 1594; and it will presently appear that his book of Algebra is of a date still earlier. If Mr Babbage's observation were historically accurate, we might claim for Napier the merit of having hit upon this ingenious device, in his unpublished work, a century before the time of Harriot. For, it will be observed, he adopts it in his chapter of equations, and defines it in these terms:—"Betwixt the parts of an equation that are equal to

each other, a double line is interposed, which is the sign of equality; thus 1 ℞=7, which is pronounced, one thing equal to seven." The fact is, however, that this notation was already more or less in use; and it is to Recorde that the merit of the first idea is due; for, in his work, first published in 1552, he says, " And to avoid the tedious repetition of these woordes, ' is equal to,' I will sette, as I doe often in woorke use, a paire of paralleles, or gemowe lines of ane length, thus =, because noe two thynges can be moare equalle."

In France, the celebrated Ramus wrote an elementary treatise on Arithmetic and Algebra, about the year 1560; but he left the science as he found it. Raphael Bombelli, whose Algebra was published at Bologna in the year 1572, in Italian, wrote more elaborately and profoundly, but did not add any thing of consequence to the labours of his predecessors. The first that can be said to have done so, between the time of Stifellius and of Napier, was Simon Stevinus of Bruges, who published La Practique d'Arithmetique about the year 1582. He afterwards put forth other works upon Arithmetic and Algebra, along with a translation of some books of Diophantus; in all of which he evinced a remarkable genius for his subject. Algebraic notation received at his hands another of those impulses by which it has so gradually reached its present

power; and he appears to have been the first who expressly mooted the doctrine of a decimal division of unit, which idea, however, remained to be developed and practically applied as a system by Napier.

A contemporary of Napier's was Vieta, whose name reflects lustre upon France. He generalized the language of Algebra, by employing letters to denote known as well as unknown quantities, and he extended the theory of equations. It is not at all likely that Napier ever saw any of his treatises, which were only first collected into one volume by Schooten, in 1646. All the other conspicuous treatises, illustrative of the progress of numerical science, are subsequent to the death of Napier. At some period of his life he had perused the work of Stevinus, as he mentions this author in the Rhabdologia; but considering how very few works of the kind existed at the time when Napier must have composed his numerical treatises, how slowly books were then spread abroad, and that literary communication between Scotland and the Continent was so slight as to leave Kepler in ignorance of the death of the Inventor of Logarithms two years after that event, it is not to be supposed that Napier had at his command even those scanty stores of information which other countries could afford, on the abstruse subjects to which he was devoted. The most

accurate chronology of his treatises, so far as can be ascertained, would seem to be between the publications of Stevinus of Germany, in 1582, and those of Vieta of France, about the commencement of the next century. Of all his predecessors and contemporaries, however, Napier was the one destined to create the greatest revolution in Numbers. This fact alone must render the perusal of his early and preparatory labours very interesting to all who are capable of appreciating his great invention. Countries the most distinguished in Europe, for men of science, had produced in that recondite path the few remarkable men so briefly and imperfectly noticed above. But even the gigantic Kepler (on his own authority we state it) had struggled in vain against the spell that yet bound the Arabic system. From Archimedes to Kepler, not one of the victorious in the field of science had struck a blow sufficient to extricate the best wing of mathematics. Until Napier arose, the Arabic scale remained undeveloped. The name of Recorde is barely sufficient to give England a place in the previous history of the progress of science; and as for Scotland, when Napier was acquiring for his country the fame of one of the greatest impulses which human knowledge has received, it was distinguished by mists which science had not penetrated, and by the Douglas wars, whose baronial leaders knew little of the denary system beyond their mail-clad hands.

Napier, in the first chapter of his Arithmetic, refers to what he terms Geometrical Logistic, as forming the subject of his third book, and to Algebra, as being the subject of a fourth book. It would appear, however, that his Algebra, so far as "orderlie sett doun," is an earlier production than either his Arithmetic or the fragment of Geometrical Logistic. This is manifest from several circumstances :—1. It is entitled "The Algebra of John Naper, Baron of Merchistoun ;" but not Liber Quartus, in correspondence with the other books. 2. Arithmetic is referred to in it ; but there is no reference to his own book of Arithmetic, as there would have been, according to his practice throughout the rest of the manuscript, had that been previously composed. 3. The treatise of Algebra is itself divided into two books ; and, while there is a systematic reference to its component parts, there is no reference to any previous books. 4. Napier adopts in his Algebra the nomination and notation which had been introduced before his time ; whereas, in his Arithmetic and Geometrical Logistic, he adopts and expounds a peculiar nomination and notation of his own, applicable to the arithmetic of Surds, and whereby he proposes to supersede that with which he operates in his treatise of Algebra. There can be little doubt, therefore, that his Algebra is a work of still earlier date than the other books ; and these, as has been seen, were prior in date to his conception

of the Logarithms, which was some time before the year 1594. Yet there is no appearance of crude or hasty composition in his earliest work. It is stamped with the same characteristics—clear and simple exposition, profound and original views, and symmetrical arrangement—which eminently distinguish all his compositions.

Having ascertained the most probable chronology of the treatises now printed, as being the occupations of a period embracing a course of years prior to 1594, we must compare the fact with Napier's time of life. He was born at Merchiston in the year 1550; so that he was only of age in 1571, the year of his marriage in Scotland. Even at this early period of his life, he must have been far advanced in his abstruse studies, in order to have obtained that complete command of Logistic,—a term he uses to denote the whole analytic art,—which had enabled him, prior to the year 1594, to digest every department of numerical science into a more comprehensive and perfect institute of the subject than had yet appeared. Napier himself has recorded an anecdote which illustrates the extraordinary precocity of his genius; and, while referring to it, I may add a few particulars relative to his early studies. It appears, from the original records of the University of St Andrews, that Napier was incorporated in the College

of St Salvator in the year 1563. But there is an earlier and very curious notice, in reference to his education. His mother's brother was that celebrated character, Adam Bothwell, the first reformed Bishop of Orkney; the same who performed the marriage ceremony between Queen Mary and the Earl of Bothwell, and who afterwards anointed and crowned the infant James VI. This prelate concludes a letter (still preserved in the Napier charter-chest) on the subject of his own private affairs, addressed "To his Bruder the Laird off Merchistoun in Loudeanne," Napier's father, with the following remarkable and prophetic sentence:—

"I pray you, schir, to send your sone Jhone to the schuyllis; oyer to France or Flandaris; for he can leyr na guid at hame, nor get na proffeit in this maist perullous worlde,—that he may be savet in it,—that he may do frendis efter honour and proffeit,—as I dout not but he will: quhem, with you and the remanent of our successione, and my sister your pairte, God mot preserve eternalle. At the yairdis in Kirkwall, this v day of December, the yeir of God 1560, be

"Your Bruder at powair,

ADAME, Bischopp off Orknay."

Napier, at the date of this letter, (in which will be

observed a promiscuous use of the Roman and the Arabic notation), was ten years of age, and his public education had not commenced. His father was only about sixteen years his senior, which probably induced the Bishop to tender his advice. But it was not until 1563 that young Napier matriculated at St Andrews; and how soon his genius there impelled him to the deepest speculations, we learn from himself. In his address "To the Godly and Christian reader," prefixed to his Commentaries on the Apocalypse, published in 1593, he tells us:—"In my tender yeares and barneage in Sanct Androis, at the schooles, having, on the one part, contracted a loving familiaritie with a certaine gentleman, a Papist,—and, on the other part, being attentive to the sermons of that worthy man of God, Maister Christopher Goodman, teaching upon the Apocalyps, I was so moved in admiration against the blindness of Papists, that could not most evidently see their seven-hilled citie Rome painted out there so lively by Saint John, as the mother of all spiritual whoredom, that not onely burstit I out in continual reasoning against my said familiar, but also, from thenceforth, I determined with myself (by the assistance of God's Spirit) to employ my studie and diligence to search out the remanent mysteries of that holy book,—as to this houre (praised be the Lorde!) I have bin doing at al such times as conveniently I

might have occasion." It is a curious trait of the early power of his mind, that, when only fourteen years of age, he should have listened so intensely to an exposition of the Apocalypse from the pulpit, bursting forth afterwards in disputation with his papistical friend and companion, until he conceived the daring project of leaving not a mystery of prophecy undiscovered,—a project eventually realized, as he supposed, by those profound but fruitless speculations, in which he has been followed by Mede, Sir Isaac Newton, and a host of moderns, who have added nothing to his labours. The anecdote not only proves the precocity of his genius, but indicates the immediate cause of his very early attraction to that profound study of numerical science, which must have been co-extensive with the progress of his very learned and long laboured work, the "Plain Discovery of the whole Revelation of St John." His varied illustrations of such propositions as, for instance, that "the forty-two months, a thousand and two hundred and threescore prophetical days, three great days and a half, and a time, times, and half a time, mentioned in Daniel and the Revelation, are all one date," prove that he was deeply versed in ancient chronology and numbers. Moreover, in attempting to expound the mystery of the name and number 666, he evinces a knowledge of the numeral system of the Greeks, and that they worked arithmetically with the letters of

their alphabet, instead of the Arabic or Indian notation now in use. It must be observed also, that the arrangement and structure of his theological work is mathematical.

There is reason to believe that Napier had made some researches abroad, relative to the history of the Arabic notation, when a very young man. In his Memoirs, I had ventured the hypothesis that he must have been abroad for a time after his studies at St Andrews. This was founded on the facts, that, as appears from the records of the University, he did not remain at St Salvator's long enough to take a degree, or to complete the usual course of studies there; and that, from the terms of a letter of the Bishop of Orkney to the Laird of Merchiston in the year 1668, young Napier seems to have been absent from home of that date. Besides, the almost invariable custom of Scotland then was, for all young men of family, having any pretensions to a learned education, to complete their studies on the Continent. Napier's Commentaries on the Apocalypse prove that he was a most accomplished scholar, that he possessed a general knowledge of the arts and sciences, and a great command of the learned and continental languages. Under these circumstances, it is more difficult to believe that he had never quitted his own rude country, than that he had obtained the usual advantage of spending some years of his

youth in France, Italy, and the Low Countries; more especially, as this very plan of his education had been earnestly recommended by his uncle, the Bishop of Orkney. In coming to this natural and indeed inevitable conclusion, when writing his Memoirs, I was not aware of a very interesting fact, subsequently mentioned to the Royal Asiatic Society by the Right Honourable Sir Alexander Johnston, (a lineal descendant of the Inventor of Logarithms,) as chairman of the Committee of Correspondence. At the anniversary meeting of the Society held on the 9th of May, 1835, Sir Alexander, in the course of an address, for whose varied information he received the thanks of the Society, took occasion to communicate the following facts relative to Napier, and the curious result of a projected life of him by Lord Napier, which I have extracted from the proceedings of the learned body to whom they were addressed:—

" The province of Madura again became an object of literary interest in the eighteenth century, in consequence of my grandfather, the fifth Lord Napier of Merchiston, having determined to write the life of his ancestor, John Napier of Merchiston, and to prefix to it a history of the knowledge which the people of India had of mathematics. It appearing by John Napier's papers, that he had, from the information he

obtained during his travels, adopted the opinion, that numerals had first been discovered by the college of Madura, and that they had been introduced from India by the Arabs into Spain, and into other parts of Europe, Lord Napier was anxious to examine the sources from whence John Napier had derived his information upon this subject; and, when he himself was abroad, he visited Venice and other places in Italy, in which he thought it was likely he should find an account of the information collected by the members of the Jesuit mission at Madura, upon this and other parts of Hindoo science. Having been successful in obtaining some interesting documents relative to the object of his researches, he returned to Scotland, and submitted them to the then Mr Mackenzie (afterwards Colonel Mackenzie), who had been recommended to him by Lord Seaforth, as a young man who had devoted himself to the study of mathematics. Lord Napier died before he had completed his life of John Napier; and Mr Mackenzie, whose mind had been turned to the subject of Hindoo science by Lord Napier, applied for, and obtained, through Lord Seaforth, a commission in the East India Company's Engineers on the Madras establishment, in order that he might have a favourable opportunity of prosecuting at Madura, the site of the ancient Hindoo college, his enquiries into the knowledge which the Hindoos possessed, in early days, of arithmetic, and the different branches of mathe-

matics. On Mr Mackenzie's arrival at Madras, finding that my father and mother (the latter being the daughter of his patron, Lord Napier, and then engaged in completing the life which had been commenced by her father) were stationed at Madura, where my father held a political situation of high trust under his friend Lord Macartney, he obtained leave from Lord Macartney, the then Governor of Madras, to join them. As soon as Mr Mackenzie reached Madura, he began his enquiries relative to the ancient Hindoo college of that place; and, in conjunction with my father and mother, formed the plan of reviving, under the protection of the English government, the Hindoo college. In furtherance of this plan, my father having obtained from the Nabob of Arcot, the then sovereign of the country, some deserted ruins in the jungle, about a mile from the fort of Madura, which were supposed to have been connected in former days with the proceedings of the Hindoo college, built upon them, at considerable expense, the house which has ever since been known at that place by the name of Johnston House, and which is still my property, laying out its different compartments, under the direction of Mr Mackenzie, in such a manner as might best suit the adaptation of it as a building, in which the mathematical instruction that Mr Mackenzie wished to be circulated amongst all the natives of the country might be pursued. The pillars which supported

this house were divided into six compartments, upon each of which all the diagrams were to be carved which were necessary to illustrate a course of arithmetic, geometry, mechanics, hydrostatics, optics, and astronomy, there being a building erected upon the roof, in which plane and spherical trigonometry were to be taught: two orreries were to be erected, the one illustrating the Ptolemaic, the other the Copernican, system of the universe; and lectures were to be given in Tamil, Telugu, Malayalam, and Canarese, pointing out the superior utility of the Copernican over the Ptolemaic system, and the great practical utility to which the sciences of Europe might be applied in every department of practical knowledge. Mr Mackenzie, shortly after he had finished this building for my father, was obliged to quit Madura on account of the public service, and the plan of the college was, owing to his absence, not then carried into effect."

It is to be lamented that the life of Napier, undertaken by the nobleman here mentioned, and subsequently by his accomplished daughter, never saw the light. In the preface to the Memoirs published in 1834, I had erroneously stated that Mrs Johnston's papers on the subject had perished by fire. I have since learned, from Sir Alexander himself, that the papers referred to in his address to the Asiatic Society, and

which were bound together in the form of a quarto volume, were lost at sea, on board of the Jane, Duchess of Gordon, when that vessel perished on her voyage from Ceylon, in 1809. As Francis, fifth Lord Napier, was he to whom the then Napier of Culcreuch presented the valuable manuscripts which compose the present volume, it is not unlikely that the papers, from which it appeared that John Napier had prosecuted some enquiries abroad relative to the history of the Arabic or Indian notation, had also been obtained from the family of Robert Napier. The fact that Napier had made this investigation is the more interesting, that he himself was the first who fully developed the system. He added to it the Logarithms, and the reciprocation of the scale in the working of decimal fractions. "There are two improvements," says Wallis (the friend and contemporary of Newton), "which we have added to the Algorism of the Arabs, since we received it from them; to wit, that of Decimal Fractions and that of Logarithms."

In the manuscripts now printed, and which have so fortunately escaped the fate of nearly the whole of Napier's private papers illustrative of his studies, it was not to be expected that he would allude to his researches abroad, or enter into antiquarian details relative to the origin of the present system of arithmetical notation. But that he well understood the prin-

ciple of its operation, and had also turned his mind to new inventions in notation, may be gathered from these manuscripts. In the first chapter of the second book of his Arithmetic, in which he discusses the nomination and notation of numerical quantities, he says that " every idiom supplies its own vocal nomination; but that the written names of integers are the nine significant figures,—1, 2, 3, 4, 5, 6, 7, 8, 9; and that these signify various numerical values, according to their change of place. This progress," he adds, " is from right to left; and the circle 0, which has no signification wherever it be placed, is merely used to indicate the progress of the significant figure, by representing the vacancy which that has occasioned in its progress. When the significant figure is occupying its first place, it is named," says Napier, " according to its own individual value; when it has progressed to the second place, it is named by its tenfold value; in the third place, a hundredfold; and so on by an infinite progress, each step attaching to the figure a value equal to the multiplication of the last step by ten." Such is Napier's explanation of that all-powerful though simple expedient; and none more accurate and lucid has been afforded during the two centuries and a half which have elapsed since he composed his digest of Numbers. The extreme ingenuity with which he adapted to this principle—of numerical values in geometrical progression, indicated by the progressive motion

of the same digit in space—a new set of digits of his own invention, intended for the Arithmetic of Surds, is well worthy of attention.

Napier commences what he terms Geometrical Logistic with an important principle,—his profound consideration and command of which most probably paved the way to his great invention. He views quantity and number under two different aspects or conditions; namely, either as separated into distinct parts, capable of being exactly expressed by absolute numbers, or fractions of numbers, which he terms discrete quantity and number; or, as consisting of an infinite continuity of parts, not to be so expressed by numbers; and this he terms concrete quantity or numbers. " If," says he, " 3 *a*" (by which he means 3 of any given quantity) " refer to three digital lines, thus, ——— ——— ———, it is a discrete number; but if it refer to a concrete and continuous tri-digital line, in this form, ———·———·———, it is called a concrete number; though not a proper concrete number, but only in relation to the quantity to which it refers. A proper concrete number," Napier adds, " is the root of a number whose root cannot be measured; that is, exactly and finitely expressed by any number, integral or fractional." He had already explained that the numerical quantity, generally termed the Power of a

number, but which he terms Radicatum, is that which can be reduced to unit, when divided one or more times by some other number, the number of the partitions being the index, and the dividing number the radix, or root. Euler, the great algebraical writer of the eighteenth century, had not a more profound knowledge of the properties of the relative numerical quantities, Root, Power, and Exponent, than Napier in the sixteenth century had of what he called Radix, Radicatum, and Index. Indeed, his opening of the subject of involution is less perplexing than Euler's, whose statement might leave the student at a loss to know why the square of a number is called the second power, and not the first. For Euler, after having said that a power of a number derives its rank, or particular denomination, from the number of times it is multiplied by itself, informs us, that a square is obtained by multiplying a number once by itself, and a cube, by multiplying a number twice by itself. Why, then, is a square called the second power, since the elevation is obtained by the first multiplication of the number by itself? Napier avoids all such perplexity, when he commences by saying, that the first step in the process of involution is to "multiply unit by the radix, which multiplication returns the radix itself; secondly, multiply that again by the radix, and the duplicatum (the expressive term he uses for square or second power) is produced; and so

on, according to the quality of the index, which," he says, " is determined by the number of units composing the index." Thus, whether he explain the nature of a radicated quantity, as that which a certain number reduces to simple unity by one or more divisions, or as that which is raised to its particular denomination by one or more multiplications of the same number into itself, he so brings unit into his statement of the involution or evolution, as at once to make manifest the meaning of the algebraic law, that, although the powers of a number are the products of the successive multiplications of that number into itself, yet every number, when thus taken as the root of successive powers, is considered the first power of itself.

Moreover, Napier did not fail to observe, that there are certain numbers which neither are to be obtained by the multiplication of any number whatever into itself, nor can be resolved into simple unit by division by any number whatever. Thus, the number 16 is reduced to 1 by four times dividing by the number 2. This discovers 2 as what Napier called the quadrupartient root of 16, and 16 as the quadruplicate, or fourth power of 2. But there is no number, for instance, which by involution will produce the number 10. The bipartient or square root of 9 is 3, because 3 times 3 is 9; but what is the square root of 10? In other words, what is the number

which, being multiplied by itself, produces 10? Not 3 times 3, that being only 9, nor yet the next number, 4, which gives too much, namely 16. The doctrine of fractions, indeed, enables us to express exactly numerical quantities between 3 and 4, nearer to each other than the relative quantities indicated by 3 and 4, and consequently a nearer approximation to the quantity sought. This approximation, however, will still be found to consist of terms, the one too much, and the other too little, exactly to express what is required; and a curious property it has, that these fractional terms may be brought closer and closer together, by an endless approximation, and still no number obtained which, by being multiplied into itself, will produce the precise number. Hence, numerically speaking, the number 10 (being a simple example of an infinity of numbers in the same predicament) has no root; that is, no root capable of finite expression in discrete or absolute number. "Now," says Napier, "a concrete number proper is the root of such irreduceable number, and these roots are commonly called surd and irrational."

Napier had too strong a hold of his subject to reject these latent and ineffable roots as no quantity at all, or as incapable of submission to the rules of Arithmetic, and the purposes of computation. He views them in their proper concrete charac-

ter of quantity or magnitude, rather than as number or multitude; and calls them "nomina," because susceptible, he says, rather of being named than numbered. Moreover, he had considered these quantities so profoundly as to discover all their computative properties, and fully to illustrate them under the operation of all the rules of Arithmetic relative to discrete number and quantity.

Having digested this important chapter of his system of Logistic, Napier was struck with the necessity of a new nomination and notation wherewith to work, more easily and effectually, the peculiar quantities in question. The ancient geometricians had derived their nomination of the successive powers of a number, entirely from the three local dimensions of nature,—length, breadth, and thickness. Napier, in his Commentaries on the Revelation, took those co-existing qualities as an illustration of the Trinity; but he was not satisfied with their illustration of the geometrical progression of numbers. A geometrical progression, of which we hear so much in all statements of the Logarithmic principle, is so called, because the successive numerical powers which compose it were named from the geometrical extensions—represented by a line, a surface or square, and a solid or cube. The square contains two of the local dimensions,—length and breadth; in the cube

are the whole three,—length, breadth, and thickness. In any progression of numbers, increasing from unit by a common ratio of multiplication, the second power contains two dimensions of the root, and was called by the ancient geometricians a square number, or square; so the third power, being composed of three dimensions of the root, obtained the name of a cubic number, or cube. But here the imperfect analogy became exhausted; for, while the powers of a number are infinite, extension in space admits of no other distinctions than the three above mentioned. Hence, by an expedient incongruous in idea as well as unwieldy in practice, the higher powers were designated as if repetitions of the geometrical dimensions. This will be seen from the following table, which is given by Napier in his earlier work on Algebra, p. 92.

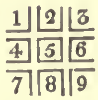
√ʒ	Radix quadrata.
√ℭ	Radix cubica.
√ʒʒ	Radix quadrati quadrata.
√ß	Radix supersolida.
√ʒℭ	Radix quadrati cubica.
√ſß	Radix secunda supersolida.
√ʒʒʒ	Radix quadrati quadrati quadrata.
√ℭℭ	Radix cubici cubica. Et sic de cæteris in infinitum.

Here the signs are just an initial abbreviation of the words, and scarcely less unmanageable in practice. Napier had sub-

sequently become sensible of the defect both of this notation and nomination. The numbers, or rather quantities, particularly dependent upon the nomenclature in question, are surds, which cannot be expressed in effable numbers, whether integers or fractions. Accordingly, in his Arithmetic (p. 11), after showing, for instance, that the surd quantity, called the cube root of 9, lurks between the numbers 2 and 3, Napier adds, "But geometricians, studious of greater accuracy, prefer the mode of prefixing the sign of the index to the radicate itself, instead of including the root between two terms. For example, they note the tripartient root of 9 thus, √ℭ9, which they pronounce the cube root of 9. I, however, note it thus, └9, and call it the tripartient root of 9; of which signs I shall more fully speak in their proper place. Hence arises geometrical or concrete numbers, commonly called irrational and surd." This was tantamount to naming the cube root of 9 (or whatever power be taken) the third root, and noting it by an equivalent numeral index; for we shall immediately see that the symbol └ was an invention of Napier's, to represent the numeral 3 used for a special purpose.

Along with a nomination expressive of the index, or quality of the root, it was Napier's object to establish a notation which, in like manner, would immediately suggest to the eye, not

merely that a surd root was taken, but what number of root it was; and this by symbols entirely new and unappropriated. Since his day, when roots were to be expressed by radical signs, instead of by the number itself evolved, the expedient devised was to retain the contracted R, as the initial of the word root, and to place within it a small numeral, as the index of the quality of the root. Thus $\sqrt[3]{10}$ expresses the cube root, or, as Napier would have called it, the tripartient or third root, of the number ten considered as a radicatum or power. This notation,—which, it will be observed, gets rid of the unwieldy expedient of repeating the initials ℨ, ℭ, and ſβ, for square, cube, and supersolid, and presents to the eye the number of evolutions requisite to obtain the particular root,—was not invented until many years after Napier's death. Now, the notation to which he refers, in the sentence quoted above, was so devised as at once to express that a root, of that number to which it was attached, was the quantity taken, the particular number or index of that root, and the fact that the root was a surd root, or concrete quantity. He took this simple combination of equal lines intersecting each other at equal intervals. In the nine compartments, which this figure presents, he inserted the nine numerals,—for the sake, he says, of assisting the memory. From the natural arrange-

ment of the figures within, it is easy to remember the number appertaining to each compartment. Separate the compartments thus, , and still the memory will retain the relation of each to the original form, even when the numerals are withdrawn. By this simple process there is actually obtained an equivalent for the nine significant digits of Arithmetic, susceptible of the same combinations, yet perfectly distinct in character. Napier's unit of this system is ⌋, and when the root which he wishes to represent is beyond the original value of the highest of his digits, which is ┌, corresponding to 9, he obtains the tenth value by moving his unit one step to the left, indicating the move by placing a circle on the right of the significant digit, thus, ⌋°. In like manner, his unit in its first place combined with the same advanced a step, thus ⌋⌋, indicates the root of the 11th division, or undecupartient root, as he named it. Thus having invented a new set of digits equivalent to 1, 2, 3, 4, 5, 6, 7, 8, 9, he submitted them to the same law of progression as in ordinary arithmetic.

Upon comparing the fragment of Geometrical Logistic with the first book of his Algebra, which Napier calls the nominate part of Algebra, it will be obvious, that the latter had

been framed before the conception of the former. Having composed two books of Algebra, he had extended his plan so as to embrace the whole art of Logistic; and the nominate part of his Algebra would probably have merged in his third book called Geometrical Logistic, while the second book of Algebra, which he called the positive or cossic part, in which unknown quantities are treated of under fictitious representations, would have formed the fourth and last book of his great digest of Numbers. It is quite consistent with the nature of the subject to suppose, that the reason why Napier's book of the Logistic of concrete quantities by concrete numbers terminates abruptly, and why no other fragment of it could be found, was, that at this stage of his speculations, he had set himself to the invention of Logarithms, which may indeed be considered as the grand result of the Logistic of concrete quantities by concrete numbers. These, in all probability, were the very speculations which led him to that achievement, of which Playfair has said, that, " at a period when the nature of series, and when every other resource of which he could avail himself, were so little known, his success argues a depth and originality of thought, which, I am persuaded, have rarely been surpassed."

Napier's device for the notation of surds is of itself suffi-

cient to prove that his invention of the Logarithms was subsequent to such speculations ; for the mind which accomplished the Logarithms, and compounded that name for them, could scarcely fail very soon to perceive that the radical signs, expressive of a surd root, were to be entirely superseded by fractional exponents. Dr Hutton committed a strange anachronism, and one which does injustice to the genius of Napier, in the following passage :—" The notation of powers and roots, by the present mode of exponents, has introduced a new and general arithmetic of exponents or powers ; for hence powers are multiplied by only adding their exponents, divided by subtracting the exponents, raised to other powers, or roots of them extracted, by multiplying or dividing the exponent by the index of the power or root. So $a^2 \times a^3 = a^5$, and $a^{\frac{1}{2}} \times a^{\frac{1}{4}} = a^{\frac{3}{4}}$; $a^5 \div a^3 = a^2$, and $a^{\frac{3}{4}} \div a^{\frac{1}{2}} = a^{\frac{1}{4}}$; the second power of a^3 is a^6, and the third root of a^6 is a^2. This algorithm of powers led the way to the invention of Logarithms, which are only the indices or exponents of powers ; and hence the addition and subtraction of Logarithms answer to the multiplication and division of numbers, while the raising of powers and the extracting of roots is effected by multiplying the logarithm by the index of the power, or dividing the logarithm by the index of the root."—Math. Dict. If the algorithm of which Dr Hutton here speaks had belonged to science before Napier invented

the Logarithms, Playfair's eulogy of him, quoted above, would not have been merited; for the arithmetic of powers and exponents might have disclosed the Logarithms even to an ordinary mathematician. The more accurate statement is, that the invention of Logarithms led the way to the algorithm of powers. Napier,—writing, be it remembered, some time in the sixteenth century, before Vieta, Harriot, Girard, and Oughtred, and when Algebra was not cultivated at all in this country,—had no assistance from the algebraic refinement of working known quantities by means of other symbols than the significant digits, or of expressing powers by means of small letters instead of numerals. He did not, for instance, consider $aaaa$ as the quadruplicatum (to use his own term) of any number a; still less did he consider the same quantity in this form, a^4; he had neither the literal notation of powers, nor the numeral notation of indices; for although, in explaining their genesis, he named the indices, one, two, three, &c., and also noted and arranged them above the powers by the numerals 1, 2, 3, &c., yet he did not systematically attach them to the root for the expression of the power. To have done so would have been to anticipate the notation of Descartes, whose epoch is 1637, twenty years after Napier's death. But, while he had not these advantages to lead him to the Logarithms, the arithmetic of exponents was so obviously to be deduced from

that invention, that, having achieved the Logarithms, he could not fail immediately to perceive the application of fractional indices for the expression of roots,—and, consequently, that his proposed notation of surds, in his Geometrical Logistic, was comparatively crude and useless. This fact, which it would occupy too much space fully to illustrate here, may be offered as internal evidence, afforded by the manuscripts now printed, that they are studies which must have preceded the conception of his great invention.

That which may truly be said to have led Napier to the invention of Logarithms, is the profound view he took of the Logistic of concrete quantity, after having thoroughly mastered that of discrete quantity. The properties of what we may now term the Logarithms of discrete ratios, namely, the series of natural numbers taken to enumerate the steps or terms of any geometrical progression represented by integers, had been known since the days of Archimedes. But the desideratum was, to prove that the infinite series of natural numbers, 1, 2, 3, 4, 5, 6, &c., might be viewed as in a geometrical progression, and then to discover the indices of all these numbers, considered as powers of some given number,—which would disclose the corresponding arithmetical series, and the Logarithms par excellence. Less profoundly considered, the

idea that the arithmetical series, or progression by equi-difference of the natural numbers, could be viewed as a geometrical series, or progression by multiplication, seemed a contradiction in terms. Napier commences by demonstrating that the progressive increase, or decrease, of a concrete or continuous quantity may be conceived to be generated by a motion in space, so regulated, in respect of the velocity, as to generate a continuous geometrical progression, of infinitely small ratios of magnitude, various terms of which might be represented, or infinitely nearly so, by the series of natural numbers, which thus all become terms of a geometrical progression. In like manner, he demonstrated the genesis of a corresponding arithmetical progression by a simultaneous motion, the velocity of which, being equal throughout and not increasing or decreasing, generated magnitudes in an arithmetical progression, which, at any given point of the progress, might be represented by a numerical expression that would serve for the index, or number of the ratios (logarithm), of the corresponding point in the simultaneous geometrical motion. His idea of motion, thus taken to generate proportional magnitudes, was analogous to the law of the Arabic notation in discrete numbers, where the significant digit may be conceived to generate a decuple progression, by travelling in a line from right to left. There are various circumstances

connected with the doctrine of Logarithms, unnecessary to be considered here, showing its natural affinity to the system of the Arabic notation, which in fact only became perfectly developed when the Logarithms were invented; and it is singular that, among the mathematical stores of those distant climes from which was derived the refined and powerful notation in question, and throughout the long ages of its operation in the hands of genius, not a trace can be discovered of a conception of that development of the scale which Napier accomplished in the Logarithms and the arithmetic of Decimal fractions. The Chinese are said to have laid claim to the invention; but the splendid copy of the Logarithms which issued from the imperial press of Pekin, contains certain errors which have been recently discovered in the European tables, previously published.

Napier's mode of demonstrating the Logarithms, by the motion of points (fluxu puncti) generating two lines—the one "when the point describing the same goeth forward equal spaces in equal times or moments," and the other (which, to facilitate his operations, he took in the decreasing ratio, having complete command of the arithmetic of negative quantities,) "when the point, describing the same in equal times, cutteth off parts continually of the same proportion to the lines from which they are cut off,"—was afterwards

adopted by Sir Isaac Newton, in similar terms, to illustrate his own discovery of Fluxions. The accidental circumstance, or first idea, which may have led any great inventor into the path of his distinction, can rarely be discovered. I have endeavoured to illustrate Napier's progress to the Logarithms through his mathematical studies now published. Newton himself, in his treatise of the Quadrature of Curves, announces the method that led him to his great discovery of Fluxions. "I consider," he says, "mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines are described, and therefore generated, not by the opposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by a continual flux; and so in other quantities. These geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles. Therefore, considering that quantities which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated, I sought a method of determining quantities from the velocities of the motion or increments

with which they are generated; and calling these velocities of the motions or increments Fluxions, and the generated quantities Fluents, I fell by degrees upon the method of Fluxions, which I have made use of here in the quadrature of curves, in the years 1665 and 1666." It is remarkable, that, instead of merely referring to the ancients in this passage, Newton had not rather said,—'And after this manner Napier, by drawing a moveable point along a right line, taught the genesis of Logarithms; and when I speak of quantities becoming greater or less according to the greater or less velocity with which the increase and decrease are generated, and of determining quantities from the velocities of the motions or increments with which they are generated,—and when I call these velocities of the motions or increments Fluxions,—I avail myself of Napier's demonstration, I adopt his language,—and even his very expressions, "fluxu," and "incrementi aut decrementi."' Newton may have closely studied Napier's published works, or he may never have seen them. The first idea is suggested by the above remarkable coincidences of thought and expression; and also by the fact that Newton's Commentaries on Scriptural prophecy is little else than a less elaborate repetition of Napier's. The other idea, however, is after all the more likely, from this circumstance, that, throughout the voluminous published works of Newton, mathematical and theological, no notice of Napier

is to be found. Yet the great lever with which Newton worked was the Logarithms; and the Binomial Theorem, says Baron Maseres, "is so very closely connected with the subject of Logarithms, as to be the foundation of the best methods of computing them."

In Napier's Arithmetic, (p. 50,) which Newton certainly never saw, there is a figurate diagram, worthy of especial notice, as being an unknown anticipation of the Arithmetical Triangle of the celebrated Blaise Pascal; presented, however, in a far more beautiful form than that of the French mathematician. The properties of this triangle are so intimately connected with the Binomial Theorem, that Bernoulli, on that account, somewhat rashly claims for Pascal the merit of the invention. "Nous avons trouvé," he says, "ce merveilleux theorème aussi bien que M. Newton, d'une manière plus simple que la sienne. Feu M. Pascal a été le premier qui l'a inventée." Maseres, who republished Pascal's mathematical works, says of them,—"These works are so full of genius and invention, that I thought I should do a service to the mathematicians of Great Britain by republishing them in this collection. Some of them, and more especially his Arithmetical Triangle, have a considerable connexion with Logarithms, by affording a good demonstration of Sir Isaac Newton's

Binomial Theorem, in the case of integral and affirmative powers, which is of great use in the construction of Logarithms." But Napier had anticipated Pascal in this invention, which, however, only received its most valuable algebraic application, dependent as that is upon the modern exponential notation, at the hands of Sir Isaac Newton.

There are other indications, in the manuscripts now submitted to the public, of an inventive genius in Numbers, far before the times in which Napier wrote. Professor Playfair, in his Dissertation on the Progress of Mathematical and Physical Science, when speaking of Albert Girard,—a Flemish mathematician, whose principal work, Invention Nouvelle en Algèbre, was printed in 1669,—observes, "He appears to have been the first who understood the use of negative roots in the solution of geometrical problems, and is the author of the figurative expression which gives to negative quantities the name of 'quantities less than nothing,'—a phrase that has been severely censured by those who forget that there are correct ideas which correct language can hardly be made to express. The same mathematician conceived the notion of imaginary roots, and showed that the number of the roots of an equation could not exceed the exponent of the highest power of the unknown quantity." Sir John Leslie, who continued Playfair's Disser-

tation, also comments upon Girard's introduction of the phraseology, 'quantities greater and less than nothing,' and upon his discovery and nomenclature of impossible quantities. It is remarkable that Playfair and Leslie, who have expressed great admiration for the genius of Napier, had not looked at or studied his published works. Napier's command of the arithmetic of + and —, the signs of positive and negative quantities, (or as he more properly phrased it, abundant or abounding, and defective quantities,) was of great assistance to him in realising his conception of Logarithms; and, in the first chapter of his Canon Mirificus, he says, (to quote from the English translation of 1616, revised by himself,) "Therefore we call the logarithms of the sines abounding, because they are always greater than nothing, and set this mark + before them, or else none; but the logarithms which are less than nothing we call defective or wanting, setting this mark — before them." Dr Horsley had fallen into the same mistake of attributing the origin of this phraseology to Girard. But as for the quantity called impossible, these authors had not the means of knowing that Napier was the person who first discovered the use of it in mathematics, and that he had somewhat exultingly recorded the fact. Having laid the foundation, (p. 19,) by an exposition of the arithmetic of + and — not inferior to Euler's, he makes the announcement, in his Logistica Geome-

trica, which I shall here translate. "Seeing, therefore, that a surd uninome may be the root either of an abounding or of a defective number, and that its index may be either even or odd, from this fourfold cause it follows, that some surds are abounding, some defective, some both abounding and defective, which I term 'gemina;' some neither abounding nor defective, which I call 'nugacia.' The foundation of this great Algebraic secret I have already laid in the sixth chapter of the first book; and, though hitherto unrevealed by any one else, so far as I know, the value of it to this art, and to Mathematics in general, shall presently appear."

There can be no doubt that by "nugacia" Napier means the impossible quantity, and that he was the very first to conceive the idea, and to propose its use, in the arithmetic of surds and the theory of equations. He explains precisely its nature, and gives rules for its notation, and applies to it the startling designations of "a quantity absurd and impossible, nonsensical and signifying nothing." He shows that all roots, whether abounding or defective, (or, as he also terms it, greater or less than nothing,) when multiplied to an even index produce an abounding radicate. Thus, take for the root +2, and let it be multiplied to the index 4, (in other words, raised to the fourth power,) and the radicate will be +16; in

like manner, raise —2 to the index 4, and the radicate will be +16; so —2 and +2 are equally the quadripartient root of 16. Hence, says he, every abounding radicate, having an even index, has two roots. These were what he called geminal. But, he adds, if roots both abounding and defective belong to the abounding radicate with an even index, then there is no root of any kind left for the same radicates having the defective sign prefixed. Hence, for example, ⊐—16 (in modern notation, $\sqrt[4]{-16}$) is an impossible quantity. The " great emolument" which Napier expected to bestow upon Mathematics by this ghost of a quantity, can only be well understood by profound mathematicians. Euler has devoted a chapter to impossible or imaginary quantities, which he concludes with these observations: " It remains for us to remove any doubt which may be entertained concerning the utility of the numbers of which we have been speaking; for those numbers being impossible, it would not be surprising if they were thought entirely useless, and the object only of an unfounded speculation. This, however, would be a mistake; for the calculation of imaginary quantities is of the greatest importance, as questions frequently arise of which we cannot immediately say whether they include any thing real and possible or not; but when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible."

But it appears, that, when Napier had broken ground upon the great field of equations, and with a vigour and command of the subject which obviously was about to anticipate the conquests of many of his illustrious successors, he was arrested in that progress by another idea. The stars were becoming too many for Tycho and Kepler; so he promised them the Logarithms. His mind had already penetrated the Arabic system in every direction. He had already orderly set down how the various operations of number and quantity gradually unfold,—how the vast fabric produces itself, growth after growth, every rule the parent of another, and the whole intimately related, in all its parts, in endless generations of Numbers. Having traced the genealogy of Numbers upwards from nothing, he had taken zero as the pivot for a reciprocal scale, and unfolded the logistic of "quantitates minores nihilo," and "quantitates impossibiles et nihil significantes." Then he had viewed unit as broken into another infinite scale, and had reduced to order all the operations of Arithmetic upon its fractions. It remained to condense the vast system into greater simplicity and power, and also to display unit as the pivot for a reciprocal play of the Arabic scale, as he had done by zero. This he accomplished by his invention and calculation of Logarithms. Moreover, in his brilliant performance of that promise to Tycho, he first applied—before either Newton or Leibnitz—

the great doctrine of Variable Quantity, and thus he not merely afforded materials for, but, to this extent, had actually anticipated, the algebraic conquests of Newton. Newton is well designed the Prince of Mathematicians. Napier is the King of Numbers.

It adds not a little to the interest with which these manuscripts must be viewed, that they had been written out for Henry Briggs, the satellite of Napier, and to whom Napier himself refers as "amico mihi longe carissimo." Briggs was ten years younger than Napier, and, about the year 1596, had been appointed Professor of Geometry, in the munificent establishment founded by Sir Thomas Gresham, where he devoted himself particularly to Astronomy, and became known to the most celebrated men of his day. In a letter to Archbishop Usher dated August 1610, Briggs says, "Concerning eclypses, you see by your own experience that good purposes may in two years be honestly crossed; and therefore, till you send me your tractate you promised the last year, do not look for much from me; for if any other business may excuse, it will serve me too. Yet I am not idle in that kind, for Kepler hath troubled all, and erected a new frame for the motions of all the seven upon a new foundation, making scarce any use of any former hypotheses; yet I dare not much blame him, save that

he is tedious and obscure, and at length coming to the point, he hath left out the principal verb—I mean his tables both of middle motion and prosthaphæreseon—reserving all, as it seemeth, to his Tab. Rudolpheus, setting down only a lame pattern in Mars; but I think I shall scarce with patience expect his next books, unless he speed himself quickly." But ere those long promised Tables were published, the Logarithms appeared, and Kepler immediately remodeled his work upon this new chapter in science. Briggs, however, was the first to catch fire at the discovery. In another letter to Usher, dated Gresham House, 10th March 1615, after speaking of the Arabic versions of the Greek philosophers, he adds, "Napper, Lord of Markinston, hath set my head and hands a-work with his new and admirable Logarithms: I hope to see him this summer, if it please God; for I never saw book which pleased me better, or made me more wonder: I purpose to discourse with him concerning eclipses, for what is there which we may not hope for at his hands?" Dr Thomas Smith, the biographer of Usher and Briggs, has painted in vivid colours the state of excitement into which the latter was thrown by the Canon Mirificus. He says that Ursin, Kepler, Frobenius, Batschius, and others, received it with great honour, but none more so than Briggs. "He cherished it as the apple of his eye; it was ever in his bosom, or his hand, or pressed to his heart; with greedy eyes, and

mind absorbed, he perused it again and again. In his study, or in his bed, his whole thoughts were bent upon illustrating it, and bringing it to the last stage of perfection; he considered that his study could not be more fruitfully, or beautifully, or gloriously bestowed than upon this most illustrious discipline; for he regarded all other work as idleness; it was the theme of his praise in familiar conversation with his friends, and ex cathedrâ he expounded it to his disciples."

Mr Hallam, in his Introduction to the Literature of Europe, recently published, has not failed to give a prominent place to Napier. But it is to be regretted that he had not derived his information, relative to Napier's works, from accurate sources; and I may take this opportunity of correcting some mistakes into which he has fallen. Speaking of the Logarithms, he says, "This Napier first published in 1614, with the title Logarithmorum Canonis Descriptio, seu Arithmeticarum Supputationum Mirabilis Abbreviatio. He died in 1618; and in a posthumous edition, entitled Mirifici Logarithmorum Canonis Descriptio, 1618, the method of construction, which had been at first withheld, is given; and the system itself, in consequence perhaps of the suggestion of his friend Briggs, underwent some change." But the real title of Napier's original publication is, "Mirifici Logarithmorum Canonis Descriptio; ejusque usus in

utrâque Trigonometriâ, ut etiam in omni Logisticâ Mathematicâ, amplissimi, facillimi, et expeditissimi Explicatio." It is of importance to preserve the true title, because it bears evidence on the face of it, that, although (probably in reference to Tycho) he specially adapted his system to Trigonometry, he was perfectly aware of its value to the whole analytic art—"in omni Logisticâ Mathematicâ." Napier died in 1617. Nor am I aware of any such posthumous edition of the Canonis Descriptio, dated in 1618. Probably this is an inaccurate reference to the distinct work of Napier's, already mentioned, published in 1619, under the joint editorship of Robert Napier and Henry Briggs, and along with which appeared a reprint of the former work. The title of the Constructio, the most beautiful of Napier's works, is,—"Mirifici Logarithmorum Canonis Constructio; et eorum ad naturales ipsorum numeros Habitudines; una cum Appendice de aliâ eâque præstantiore Logarithmorum specie condendâ. Quibus accessere Propositiones ad triangula spherica faciliore calculo resolvenda. Una cum Annotationibus aliquot doctissimi D. Henrici Briggii in eas et memoratam appendicem." In the Memoirs of Napier I had called attention to the fact, that his works were more frequently referred to than examined; and that even Playfair, Leslie, and Dr Horsley were not acquainted with the original institute of the Logarithms. M. Biot, accordingly, when

reviewing the Memoirs, had sought out the original editions of Napier's works, and he appears to have been much struck with their power. Referring to the development, and logarithmic application, of the algebraic calculus since Napier's time, he adds,—" I declare, however, to the honour of Napier, that these means produce nothing which cannot be very easily attained by his own method; and if, as it is natural to suppose, this assertion may seem to our analysts something more than rash, I hope presently to afford unanswerable proofs of its accuracy. But to form this just idea of Napier's operations, it is necessary to study his own works, especially his second work, in which he unfolds his method,—and not to rely upon extracts." In reference to the above, M. Biot prefixes to his review an investigation which he entitles, " Analyse et restitution de l'ouvrage original de Napier, intitulé, Mirifici Logarithmorum Canonis Constructio."

In the title-page of the Constructio, of which Briggs himself was one of the editors, will be found an allusion to Napier's improvement of his own system; and in the preface, the improvement is declared to have originated with Napier himself, nor is there any allusion to a suggestion from Briggs. In his letter to the Chancellor, Napier also mentions the improvement he intended, without any reference to Briggs. Indeed, the

original edition of his Canon contains an allusion to the same, before Briggs knew of the Logarithms. Mr Hallam's doubt on the subject, which he thus repeats in another passage,—"It is uncertain from which of them the change in the form of Logarithms proceeded,"—would not have occurred had he examined these works. We have the assurance both of Napier and Briggs that there is no dubiety about the matter. Indeed, Briggs has recorded the exact state of the case in an interesting statement prefixed to his Arithmetica Logarithmica, published in London in 1624, and which I shall here translate:—"That these Logarithms differ from those which that illustrious man, the Baron of Merchiston, published in his Canon Mirificus, must not surprise you. For I myself, when expounding publicly in London their doctrine to my auditors in Gresham College, remarked that it would be much more convenient that 0 should stand for the logarithm of the whole sine, as in the Canon Mirificus; but that the logarithm of the tenth part of the same whole sine, that is to say, 5 degrees, 44 minutes, and 21 seconds, should be 10,000,000,000. Concerning that matter I wrote immediately to the author himself; and, as soon as the season of the year, and the vacation-time of my public duties of instruction permitted, I took a journey to Edinburgh, where, being most hospitably received by Napier, I stuck to him for a whole month. But, as we held discourse

concerning this change in the system of Logarithms, he said that for a long time he had been sensible of the same thing, and had been anxious to accomplish it, but had published what he had already prepared, until he could construct tables more convenient, if other weighty matters, and the frail state of his health would suffer him so to do. But, he conceived, the change ought to be effected in this manner, that 0 should become the logarithm of unity, and 10,000,000,000 that of the whole sine. This I could not but admit was by far the fittest modification; so, casting aside what I had already prepared, I commenced under his advice to bend my mind to the calculation of these tables; and in the following summer I again took journey to Edinburgh, where I submitted to him the principal part of those tables which are here published; and I was about to do the same even the third summer, had it pleased God to have spared him to us so long."

It is not for the sake of asserting his right to what, at all events, is a simple derivative idea, that these proofs are referred to. It is because the dubiety expressed by Mr Hallam might infer that Napier had done injustice to Briggs, whose genius he appreciated, and whose friendship and admiration he cordially returned. "I will acquaint you," says Lilly, in his Life and Times, "with one memorable story related unto me by John

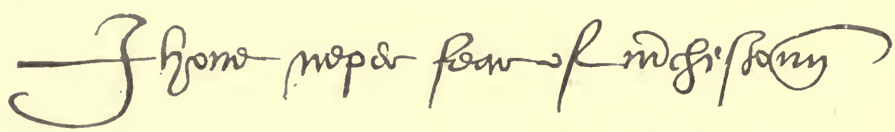
Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to James I. and Charles I. When Merchiston first published his Logarithms, Mr Briggs, then reader of the astronomy lectures at Gresham College in London, was so surprised with admiration of them, that he could have no quietness in himself until he had seen that noble person whose only invention they were. He acquaints John Marr therewith, who went into Scotland before Mr Briggs, purposely to be there when these two so learned persons should meet. Mr Briggs appoints a certain day when to meet at Edinburgh, but, failing thereof, Merchiston was fearful he would not come. It happened one day as John Marr and the Lord Napier were speaking of Mr Briggs—'Oh! John,' saith Merchiston, 'Mr Briggs will not come now.' At the very instant one knocks at the gate; John Marr hasted down, and it proved to be Mr Briggs, to his great contentment. He brings Mr Briggs into my Lord's chamber, where almost one quarter of an hour was spent, each beholding other with admiration, before one word was spoken. At last Mr Briggs began,—'My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto Astronomy, viz. the Logarithms; but, my lord, being by you found out, I wonder

nobody else found it out before, when, now being known, it appears so easy.' He was nobly entertained by the Lord Napier; and every summer after that, during the Laird's being alive, this venerable man went purposely to Scotland to visit him."

An engraving of the old Castle of Merchiston, in its primitive state, where this symposium was held, illustrates the present volume. It is as nearly as possible a fac-simile of a sketch by "Grecian Williams." The portrait of Napier is from the original belonging to the Lord Napier, and which has always been in possession of the Napier family. It is engraved now for the first time. The fac-simile of the autograph is from a letter of Napier's to his father, dated from the residence of his father-in-law, Stirling of Keir. An odd idea has arisen that he sometimes designed himself "Peer of Merchiston." In the beautiful and ably illustrated pictorial edition of the works of Shakspere, now publishing, the following ingenious note is founded on the error:—"A remarkable illustration of our belief, that Peer and Fere were cognate terms, and that a Fere or Fear was one holding of the Crown in Fee, is furnished by the title which the famous John Napier attached to his name. At the end of the dedication to his 'Plain Discovery of the whole Revelation of St John,' in the edition of 1645, Napier

signs himself ' Peer of Merchiston.' Mr Mark Napier, in the Life of his great ancestor (1834), says, that the true signature is Fear of Merchiston, and that Fear means that he was invested with the Fee of his paternal barony. Peer might have been a printer's or transcriber's substitution for Fear, or Fear might have been rejected by Napier for the more common word Peer."—(Illust. Henry IV., Act I.)

But there can be no doubt, that, in the instance relied upon, Peer was simply a misprint for Feer. That Napier had so written his signature for the edition of 1645, the only instance of its occurrence, is put out of the question by the fact, that, before that date, he had been dead for eight-and-twenty years. The following is a perfect fac-simile of his signature to a lease, dated at his place of Gartnes, 23d April, 1584 :—



MARK NAPIER.

11, Stafford Street,
November 1, 1839.

DE ARTE LOGISTICA.

THE
BARON OF MERCHISTON
HIS BOOKE OF ARITHMETICKE
AND ALGEBRA.

FOR MR HENRIE BRIGGS

PROFESSOR OF GEOMETRIE

AT OXFORDE.

LIBER PRIMUS.

DE COMPUTATIONIBUS QUANTITATUM OMNIBUS LOGISTICÆ SPECIEBUS COMMUNIUM.

CAPUT I.

DE COMPUTATIONIBUS PRIMIS.

Logistica est ars bene computandi.

Computatio est actio seu operatio quæ ex pluribus quantitatibus, et quantitatum proprietatibus datis, quæsita invenit.

Dantur autem aut vocali nominatione, aut graphicâ notatione.

Unde in omni Logisticâ primò procedunt nominatio et notatio; mox cum eis succedit computatio.

Computatio autem est simplex vel composita.

Simplex est computatio, quæ ex duabus datis tertiam unicâ aut unimodâ operatione invenit.

Simplex computatio vel est prima vel orta.

Prima est computatio, quæ quantitatem cum quantitate semel tantum computat.

> Atque hæc, ex totius, partis, et residuæ, duabus quibuscunque datis, tertiam quamcunque invenit: Quod mox patebit exemplis subsequentibus.

Hæc autem vel est additio vel substractio.

ADDITIO est computatio prima quâ plures quantitates adduntur, et producitur tota.

> Exempli gratiâ, addantur 3 et 4, et producentur 7 pro totâ; item addantur 2, 3, et 4, et producentur 9 pro totâ.

SUBSTRACTIO est computatio prima quâ substrahendum à minuendo aufertur, et producitur residuum.

> Ut auferendo 4 à 9 remanent 5. Dicuntur autem 4 auferendum, 9 minuendum, et 5 residuum. Sic ablatis 3 à 5 remanent per substractionem 2.

Substractio aut est æqualium, et nihil remanet, aut inæqualium.

Inæqualium verò est aut quantitatis minoris à majore, et remanet quantitas major nihilo, aut quantitatis majoris à minore, et residuum erit minus nihilo.

> Ut substractis 5 ex 5 remanet nihil; substractis 3 à 5 remanent 2, majores quidem nihilo; substractis autem 7 à 5 relinquentur 2 minores nihilo, seu nihil minutum duobus.

Ex his ergo constat defectivas quantitates hinc originem trahere, ex substractione nimirum majoris à minore: De quibus suo loco agetur.

Ex præmissis clarum est additionem et substractionem relata esse; atque ideo alteram alterius examen.

Examina enim definimus ea tantummodo quæ tum omnibus tum solis recte computatis conveniunt.

> Ut supra, pro examine substractionis, an 3 substracta ex 5 relinquant 2, adde 2 et 3 et restituent 5. Et contra, pro examine additionis, an 2 et 3 faciant 5, substrahe 3 ex 5, et prodibunt 2 restituta; vel aliter, substrahe 2 ex 5, et prodibunt 3 ut prius.

Est et præter hæc aliud examen substractionis in se, substrahendo nimirum residuum ex minuendo ut relinquatur prius substrahendum.

> Ut pro examine an 3 substracta ex 5 relinquant 2, substrahe 2 ex 5, et restituentur 3.

Habes itaque, ex totius, partis, et residuæ, duabus quibuscunque datis, tertiam, per additionem et substractionem.

CAPUT II.

DE COMPUTATIONIBUS ORTIS EX IPSIS PRIMIS.

Hucusque de primis computationibus egimus; sequuntur ortæ à primis.

Ortæ sunt quæ quantitatem cum quantitate pluries computant. Atque hæ ex prioribus aliquoties continuatis naturalem originem ducunt.

Ortæ item vel ex primis, vel ex primò ortis continuatione oriuntur.

Ortæ ex primis sunt, quæ ex totius, partis, et partem cognominantis. duabus quibuscunque datis tertiam inveniunt. Exemplis mox patebunt hæc.

Ortæ autem ex primis sunt multiplicatio ex continuatâ additione, et partitio ex continuatâ substractione.

Est ergo multiplicatio, alterutrius datarum toties continuata additio quoties est in alterâ unitas; et quod producitur multiplum dicitur.

> Ut multiplicare 3 per 5 est alterutrius v.g. trium quinquies continuata additio, quæ 15 efficit; vel quinarii ter additio, quæ totidem etiam efficit. Et horum 3 et 5, alterum multiplicans, et alterum multiplicandum, producta verò 15 multiplum dicuntur.

In his se habet unitas ad multiplicantis et multiplicandi alterutrum, ut alterum ad multiplum.

> Ut in superiore exemplo, ita se habet 1 ad 3, ut 5 ad 15 ; seu ita 1 ad 5, ut 3 ad 15.

Multiplicationis species infinitæ sunt; ut duplatio, quæ est multiplicatio quantitatis oblatæ per 2 ; triplatio, quæ per 3 ; quadruplatio, quæ per 4 ; et ita deinceps.

In his multiplicanda dicuntur, duplandum, triplandum, quadruplandum, &c. ; multiplicantia sunt 2, 3, 4, &c. ; multipla—dupla, tripla, et quadrupla, &c. dicuntur.

Partitio est partientis à partiendo substractio in nihilum usque continuata ; et numerus substractionum est quotus quæsitus.

> Ut sint partienda 15 per 5, auferantur 5 ex 15 continuâ substractione donec nihil remanserit, et fient substractiones numero tres : 3 ergo sunt quotus quæsitus, 15 sunt partiendum, et 5 partiens.

In his se habet unitas ad quoti et partientis alterutrum, ut alterum ad partiendum.

> Ut superiore exemplo, ita se habet 1 ad 3, ut 5 ad 15 ; vel 1 ad 5, ut 3 ad 15.

Partitio aut est æqualium, et producitur unitas, aut inæqualium.

Inæqualium verò aut est minoris per majorem, et quotus est fractio, seu fracta quantitas unitate minor ; aut majoris per minorem, et fit quotus unitate major.

> Ut sint partienda 10 per 10, producitur præcise unitas ; at 10 divisis per 13, producuntur decem decimætertiæ partes unitatis

et unitate minores; tertiò, partitis 10 per 5, producuntur 2 unitate majores.

Partitio rursus majoris per minorem, aut est perfecta aut imperfecta. Perfecta, ubi nullæ sunt reliquiæ. In his quotus est integrorum.

Ut superiore exemplo partitis 10 per 5, producuntur 2, præcise absque reliquiis.

Imperfecta, verò, quæ reliquias impartitas relinquit.

Ut si partienda offerantur 16 per 5, et inde produxeris 3, et unitatem indivisam remanere permiseris, imperfecta dicetur partitio.

Ex his ergo constat tam ex partitione minoris per majorem, quàm ex imperfectâ partitione majoris per minorem, fractiones originem trahere: De quibus suo loco.

In partitione, quantitas quæ partienda offertur partiendum dicitur; datarum altera, partiens aut partitor; quæ provenit quotus dicitur; et si quid indivisum remanserit, reliquiæ.

Ut præcedente exemplo, 16 dicuntur partiendum, 5 partiens, 3 quotus, et unitas superstans reliquiæ.

Partitionis species infinitæ sunt; ut bipartitio, quæ est partitio quantitatis oblatæ in duo æqualia; tripartitio, quæ in tria; quadripartitio, quæ in quatuor; et ita deinceps.

In his partienda dicuntur bipartiendum, tripartiendum, quadripartiendum, &c.; partientes sunt 2, 3, 4, &c.; et quoti dicuntur partes dimidia, tertia, quarta, &c.

Ex præmissis constat multiplicationem et perfectam partitionem relata esse, atque alteram alterius examen.

Ut si dubites an 3 multiplicata in 5 producant 15, pro examine partire 15 per 5, et cum inde redeant 3, scis te recte multiplicâsse: vel aliter, partire 15 per 3 et redeant 5, ut prius. Item, si dubites an 16 divisa per 5 producant 3, relictâ unitate indivisâ, multiplicato 3 in 5, non redeunt 16, sed 15; ideo addenda erit unitas examini, et partitio arguetur unitate imperfecta esse.

Est et præter hæc aliud partitionis examen in se, nimirum partiendo partiendum per quotum, ut pristinus inde redeat partitor.

Ut pro examine an 15 partita per 5 producant 3, partire 15 per 3, et redibunt 5, ut prius.

Habes itaque ex totius, partis, et partem cognominantis, duabus quibuscunque datis, tertiam per multiplicationem et partitionem.

CAPUT III.

DE COMPUTATIONIBUS ORTIS EX PRIMO ORTIS: RADICALIS MULTIPLICATIO ET PARTITIO.

Hactenus ortæ ex ipsis primis; sequuntur ortæ ex primò ortis.

Ortæ ex primò ortis sunt computationes quæ, ex radicati, indicis, et radicis, duabus quibuscunque datis, tertiam inveniunt.

Radicatum est quod aliquoties partitum per quantitatem aliquam in unitatem redit; et quotus ille partitionum index dicitur; quantitas autem partiens est ipsa radix.

Ut ex his tribus terminis, 32, 5, et 2, 32 dicuntur radicatum, quia ea partita quinquies per 2 in unitatem redeunt; scilicet

primâ partitione fiunt 16, secundâ 8, tertiâ 4, quartâ 2, quintâ denique 1 : Atque igitur horum 5 sunt index, et 2 sunt radix.

Ortæ autem ex primò ortis aut sunt radicalis multiplicatio ex continuatâ multiplicatione ; aut radicalis partitio, et radicis extractio, ex continuatâ partitione.

Radicalis Multiplicatio est radicis oblatæ toties continuata multiplicatio quoties est in indice unitas ; et producitur radicatum quæsitum.

Ut dicuntur 2 radicaliter multiplicari ter, cum 8 inde producuntur, quia ternarius index tres unitates continet ; et ex primâ ab unitate multiplicatione fiunt 2, secundâ 4, tertiâ 8, quæ quidem sunt radicatum quæsitum.

Radicalis multiplicationis species infinitæ sunt : Ut duplicatio, quæ est multiplicatio duorum æqualium invicem, aut datæ bis positæ ; triplicatio, quæ est datæ ter positæ, aut trium æqualium datæ.

In his radicata dicuntur duplicatum, triplicatum, quadruplicatum : Indices,—duo, tria, quatuor : Radices,—bipartiens, tripartiens, quadripartiens.

Ut posito binario pro radice, et binario pro indice, bini binarii faciunt 4 duplicatum : Et indice ternario totidem bina, scilicet bis duo bina efficiunt 8 triplicatum : Sic indice quaternario totidem bina, scilicet bis duo bina bis faciunt 16 quadruplicatum. Et ita in infinitum posito binario pro radice, ut in sequente tabellâ, cujus prior series est indicum, posterior radicatorum.

	I.	II.	III.	IIII.	V.	VI.	VII.	&c.
1.	2.	4.	8.	16.	32.	64.	128.	&c.

Radicalis Partitio est radicati per radicem partitio in unitatem usque continuata, et numerus partitionum est index quæsitus.

Ut radicatum 8 partitum per radicem 2, usque in unitatem, facit primâ partitione 4, secundâ partitione 2, et tertiâ 1. Unde ternarius, numerus scilicet partitionum, est index quæsitus.

Hìc indicum et radicum species sunt, ut supra in multiplicatione radicali.

CAPUT IV.

DE RADICALI EXTRACTIONE.

Radicis extractio, dato indice, est inventio quantitatis quæ datum radicatum radicali multiplicatione restituit; idemque radicali partitione dividit.

Ut radicati 8 si quæratur radix tripartiens, ea, per regulas suo loco tradendas, invenietur esse 2; binarius enim radicali triplicatione restituit primò 2, secundò 4, tertiò 8 radicatum: Idemque radicatum contra radicali partitione primò in 4, deinde in 2, tertiò in unum redigit.

Radicis extractio aut est perfecta aut imperfecta.

Perfecta, ubi nullæ supersunt reliquiæ.

Ut in superiore exemplo.

Imperfecta verò, ubi aliquæ supersunt reliquiæ irresolubiles.

Ut si radix tripartiens fuerit extrahenda ex radicato 9, ea quàm proxime erit binarius, qui radicaliter triplicatus restituit 8, et non 9; relictâ ergo unitate non extractâ, imperfecta dicitur extractio.

Hîc etiam radicati, radicis, et indicis species sunt, ut supra in multiplicatione radicali. Et quæ post extractionem remanserint reliquiæ irresolubiles dicuntur.

Quod ex imperfectâ extractione provenit est minor terminus, cui si unitatem adjeceris erit major terminus, inter quos vera et perfecta continetur et latet radix.

> Ut in superiore exemplo provenit binarius radix imperfecta novenarii, cui si unitatem adjeceris, fiet ternarius, inter quos latet vera et perfecta tripartiens radix novenarii.

Verùm Geometræ, majoris accurationis studiosi, ipsum radicatum signo indicis prænotare malunt, quam radicem inter terminos includere.

> Ut in superiore exemplo radicem tripartientem novenarii ita notant, √ ℭ 9 : quam radicem cubicam novenarii pronuntiant. Nos autem sic notamus, ⌊9, et radicem tripartientem novenarii appellamus : De quibus signis amplius suo loco dicemus.

Hîc numeri Geometrici seu concreti, quos irrationales et surdos vocant, ortum habent.

In his computationibus radicalibus, indices alii sunt pares, alii impares, alii rursus primi, id est unitate solâ dividui, alii compositi, id est numero aliquo perfecte dividui.

> Ut indices 2, 4, 6, pares sunt ; 3, 5, 7, impares : Indices autem 2, 3, 5, 7, 11, primi sunt, nec ullo numero dividui : 4, autem, 6, 8, 9, 10, &c. sunt compositi ex numeris ; nimirum 4 ex duobus binariis ; 6 ex duobus ternariis, aut tribus binariis ; 8 ex duobus quaternariis, aut quatuor binariis ; 9 ex tribus ternariis ; 10 ex duobus quinariis.

Hinc fit radicalis multiplicationis atque extractionis compendium ubi indices sunt compositi, facilius enim per componentes sigillatim multiplicantur, aut extrahuntur, quàm per compositos.

Exempli gratiâ, radix quadripartiens difficilius uno opere extrahitur ex radicato dato, quàm si ejusdem radicem bipartientem primò extraxeris atque hinc deinde aliam bipartientem. Sic sextuplicare radicem, vel sextupartientem radicem extrahere, non tam facile sit quàm si primò triplicaveris, aut tripartientem extraxeris, inde duplicaveris aut bipartientem extraxeris. Similis est similium ratio. Exemplum in numeris : Radix sextupartiens è 64 facilius extrahitur primò radicem tripartientem extrahendo, ut fiant 4, inde bipartientem extrahendo, ut fiant 2 ; vel primò radicem bipartientem extrahendo è 64, ut fiant 8, inde extrahendo radicem tripartientem è 8, ut fiant 2, radix sextupartiens quæsita.

Ex præmissis colligitur radicalis multiplicationis, partitionis, et extractionis singulas, duo habere examina ; nimirum multiplicatio probatur vel partitione vel extractione ; partitio probatur vel multiplicatione vel extractione ; extractio, vel multiplicatione vel partitione.

Ut in superiore exemplo radicati 32, indicis 5, et radicis 2, si dubites an 32 sint quintuplicatum binarii, radicaliter partire 32 per 2 et incides in 5, indicem pristinum ; vel extrahe radicem quintupartientem 32 et incides in 2, pristinam radicem ; unde arguitur 32 esse radicatum verum : Item si dubites an 2 sint radix, per eam divide radicaliter 32, et incides in indicem 5, aut eundem binarium quintuplicato, et incides in 32 : Denique ut probetur an 5 sint verus index, extrahe radicem quintupartientem è 32, et incides in 2, aut radicaliter quintuplicato 2, et incides in 32.

Habes itaque ex radicati, indicis, et radicis, duabus quibuscunque datis, tertiam per radicales multiplicationem, partitionem, et extractionem.

CAPUT V.

DE COMPUTATIONIBUS COMPOSITIS.

Hucusque computationes simplices; sequuntur compositæ seu regulæ.

Composita est computatio quæ ex pluribus quantitatibus datis, atque pluribus et diversimodis operationibus, quæsitam producit.

Compositæ computationes, seu regulæ, vel sunt proportionalium, vel disproportionalium.

Regulæ proportionalium sunt, quæ per solas computationes simplices proportionatas, scilicet multiplicationes et partitiones, quantitatem quæsitam ex pluribus datis inveniunt.

> Ut si quæratur, is qui tribus horis quatuor miliaria incedit, sex horis quot miliaria incedet? Item si sex boves nutriantur tribus mensuris fœni quatuor diebus, quæraturque quot boves nutriri possunt quinque mensuris fœni duobus diebus? Item 20 solidi Scotiæ sunt una libra, 2 libræ sunt tres marcæ, 5 marcæ valent unicum coronatum; quot ergo solidos valebunt 9 coronati? Quæstiones proportionalium sunt absque ullâ additionum vel substractionum introductione. Multiplicationes enim et partitiones sunt per consectaria suarum definitionum proportionales.

In his spectantur situs et operatio.

Situs quatuor præcepta sunt.

Primum, ut ductâ lineâ, quantitati quæsitæ cum suis collateralibus præparetur sub lineâ locus.

Ut in exemplis superius propositis subsequitur.

Pro secundo et tertio præcepto, promptior et facilior esset modus, subjungere lineæ numeros minores datos una cum quantitate quæsitâ; sed an is modus sit tutior nondum liquet.

Primum exemplum.

6 horæ, 4 miliaria ;
3 horæ, quæsita miliaria.

Secundum exemplum.

6 bo. 5 mens. 4 dieb.
quot bo. 3 mens. 2 dieb.

Tertium exemplum.

20 sol. 2 lib. 5 marc. 9 coron.
quot sol. 1 lib. 3 marc. 1 coron.

Secundum, ut duæ quantitates, quarum alterâ crescente altera decrescit, ex eodem latere lineæ collaterales statuantur.

Ut in primo exemplo, quanto plures fuerint priores horæ, tres nempe, tanto pauciora erunt miliaria quæsita. Sic, crescente numero boum (ut secundo exemplo), decrescit numerus dierum quibus eodem pabulo nutriantur. Unde 3 horæ et quæsita miliaria, atque 6 horæ et 4 miliaria; itemque 6 boves et 4 dies, atque quæsiti boves et 2 dies, in eodem latere linearum collocantur.

Tertium, ut duæ quantitates, simul crescentes vel simul decrescentes, ex adversis lineæ lateribus statuantur.

Ut crescentibus 3 horis, crescere etiam 4 miliaria erit necesse, et contra. Sic crescentibus 6 horis, crescere miliaria quæsita,

et contra. Item aucto numero boum, augeri et eorum pabulum necesse est, et contra, minuto minui. Sic auctis diebus augeri fœnum, et minutis minui. Tertio exemplo, auctis vel minutis solidis, simul et libras priores eis æquales augeri vel minui est necesse. Sic cum posterioribus libris, 2 scilicet marcæ priores 3 eis æquales crescent et decrescent; atque cum marcis 5 posterioribus coronatum unum. Tandemque cum coronatis 9 posterioribus, quæsitum solidorum numerum simul augeri et minui necesse est. Unde ex singulis hisce binis altera quantitas sub, et altera supra lineam constituitur, ut superius cernere licet.

Quartum, ut binæ quantitates cognomines lineâ illâ semper sejungantur.

Prout in superioribus exemplis, 3 horæ à 6 horis, et 4 miliaria à miliaribus quæsitis in primo exemplo: Et 6 boves à bobus quæsitis, 5 mensuræ fœni à 3 mensuris, et 4 dies à 2 diebus in secundo exemplo: Et in tertio exemplo 20 solidi à solidis quæsitis, et 2 libræ ab 1 librâ, et 5 marcæ à 3 marcis, et 9 coronati ab 1 corònato, interpositâ lineâ disjunguntur.

His observatis, ad omnium ejusmodi quæstionum solutionem unicum inserviet generale hoc operationis præceptum:

Multiplica quantitates superiores invicem, item et inferiores invicem, deinde multiplum superiorum partire per multiplum inferiorum, et quotus erit quæsitum quæstioni satisfaciens.

Ut in primo exemplo, multiplica superiores 6 et 4 invicem, fient 24; quæ partire per inferiorem numerum 3, et fient 8, numerus miliarium quæsitus. In secundo exemplo, multiplica superiores 6, 5, et 4 invicem, et fient 120; inde multiplica 3 per 2, fient 6; per quæ partire 120, producentur 20, numerus

boum satisfaciens secundæ quæstioni. Item in tertiâ quæstione multiplica superiores 20, 2, 5, et 9 invicem, fient 1800; deinde multiplica 1, 3, et 1 invicem, fient tantum 3; per quæ partire 1800, fient 600, numerus solidorum valentium 9 coronatos.

Itaque omnes species regularum proportionalium unicâ generali methodo, et operatione, comprehendimus.

De hâc doctrinâ infinitas,—ut regulæ trium seu aureæ, simplicis, duplicis, quinque quantitatum, sex quantitatum, directæ, inversæ, &c.,—species et formas tradunt authores; nec tamen triplices, aut alias ejus multiplices formas attigerunt, quas omnes hîc breviter habes.

Atque hæ sunt proportionalium; sequerentur disproportionalium regulæ: Sed quia hæ, præter computationes proportionatas, additiones etiam et substractiones, et alias computationes proportionem disturbantes immistas habent, has ideo omnes missas facimus, quod unica pro eis omnibus inserviet nobis Algebra.

Ut sunt potissima pars omnium arithmeticarum regularum, alligationis, societatis, falsi, simpli, dupli, et aliarum plurimarum, itemque Geometricarum propositionum, problematum, theorematum, &c. quæ, confusa tum varietate tum multitudine, memoriam disturbant;—has ergo relinquimus, Algebram tractaturi.

CAPUT VI.

DE QUANTITATIBUS ABUNDANTIBUS ET DEFECTIVIS.

Hactenus computationes quantitatum in genere ; sequuntur suarum specierum.

Primò, ergo, quantitates aut sunt abundantes, aut defectivæ.

Abundantes sunt quantitates majores nihilo, et augmentum præ se ferunt.

Hæ, aut nullo, aut hoc signo +, quod copula augmenti dicitur, prænotantur.

> Ut si nihil debentis opes æstimentur 100 coronatorum ; eæ aut sic, 100 cor., aut sic, +100 cor., prænotantur ; et sic pronuntiantur, auctæ centum coronatis, commodum semper et lucrum significando.

Harum computationes tam ex præmissis quàm subsequentibus habentur.

Defectivæ sunt quantitates minores nihilo, et minutionem præ se ferunt.

Hæ, semper hoc signo —, quod minutionis copula dicitur, prænotantur.

> Ut si ejus opes æstimentur cujus debita excedunt bona 100 coronatis, merito ejus opes sic prænotantur, —100 coronatis ; et sic pronuntiantur, minutæ centum coronatis, damnum semper et defectum significando.

Defectivarum ortum et originem superius ex substractione majoris à minore provenire ostendimus.

Adduntur abundantes et defectivæ, si copulæ sunt similes, aggregato eorum præponendo communem copulam.

Ut addendo +3 et +2, fient +5: Item addendo —4 et —6, fiunt —10.

Adduntur verò, si copulæ sunt dissimiles, eorum differentiæ præponendo copulam majoris quantitatis.

Ut addendo +6 et —4, fiunt +2: Sic —6 et +4 addita, fiunt —2, copulam majoris, scilicet senarii, semper præponendo differentiæ.

Substrahuntur autem, si substrahendi copulam mutaveris, eamque ad alteram datarum addideris per præcedentes regulas.

Ut sint substrahenda +5 ex +8; muta +5 in —5, et per præmissam adde —5 ad +8, et fiunt +3 pro residuo substractionis quæsito: Item sint ex —5 substrahenda +8; muta +8 in —8, et ea adde ad —5, et fiunt —13, residuum quæsitum: Sic substrahendo —5 ex +8, fiunt +13; et +5 ex —8, fiunt —13; et —5 ex —8, fiunt —3; et +8 ex +5, fiunt —3; et —8 ex +5, fiunt +13; et —8 ex —5, fiunt +3.

Abundantes et defectivæ multiplicantur et partiuntur, si copulæ sint similes, præponendo multiplo vel quoto copulam pluris; et si copulæ sint dissimiles, prænotando copulam minutionis.

Ut sint multiplicanda +3 per +2, aut —3 per —2, producitur multiplum +6; et si dividenda sint +6 per +3, vel —6 per —3, producitur quotus +2; si verò +3 per —2, aut —3 per +2 multiplicaveris, producentur —6, multiplum quæsitum; et si diviseris +6 per —3, aut —6 per +3, producetur quotus —2.

Radices, tam abundantes quàm defectivæ, pari indice multiplicatæ, producunt radicatum abundans.

Ut sit radix +2, quam ad indicem 4 multiplicabis, et fient primò +2, secundò +4, tertiò +8, quartò +16 abundans; similiter, —2 multiplicata facient primò —2, secundò +4, tertiò —8, quartò item abundans +16 ut suprà.

Hinc sequitur, radicati abundantis indice pari duas esse radices, alteram abundantem, alteram defectivam; deficientis verò radicati, nullam.

Ut superiore exemplo radicati +16 abundantis, tam abundans +2, quàm defectiva —2, erant radices quadripartientes, ut ex superioribus et utriusque examine patebit. Unde nulla restat, sive abundans sive defectiva, quæ sit radix quadripartiens defectivæ —16.

Radices abundantes indice impari reddunt (multiplicatione radicali) radicata abundantia, et defectivæ, defectiva.

Ut radix abundans +2, indice impari 5, radicaliter multiplicata reddit +32; scilicet, primò +2, secundò +4, tertiò +8, quartò +16, quintò +32, radicatum abundans. Sic radix deficiens —2, indice 5, radicaliter multiplicata facit —32; scilicet, primò —2, secundò +4, tertiò —8, quartò +16, quintò denique —32, radicatum defectivum dictæ radicis.

Simili modo hinc sequitur, quod radicatum impari indice radicem habeat unicam tantum; abundans, abundantem; et defectivum, defectivam.

Ut superiore exemplo radicatum abundans +32, indice 5, habebit radicem abundantem +2. Sic radicatum defectivum

—32, indice eodem, habebit radicem defectivam —2, quod ex præcedente exemplo, et utriusque examine liquidò constat.

Regula proportionis non est hìc repetenda, quod ea ex multiplicationibus et partitionibus componatur, et per præmissa acquiratur.

CAPUT VII.

DE QUANTITATIBUS FRACTIS.

HACTENUS prima quantitatum divisio ; sequitur secunda.

Secundò, etiam, quantitates aut sunt integræ, aut fractæ.

Quantitates integras hìc dicimus, quæ aut unitatem, aut nullum habent denominatorem. Integrarum autem per se computationes ex præmissis, cum fractis, verò, ex sequentibus habemus.

Fractas verò dicimus, quæ denominatorem diversum ab unitate habent numeratori suppositum.

Denominator est quantitas supposita lineæ, quæ per quot partes dividenda sit tota indicat.

Numerator autem est quantitas superposita lineæ, quæ quot ex illis partibus sumendæ sint denotat.

> V. g. Hæc quantitas, $3\,ab$, est integra quantitas. Sic (quod idem est) $\frac{3\,ab}{1}$ est etiam integra, sub specie tamen fractionis. Item $\frac{3\,ab}{2\,bc}$, et $\frac{5\,a}{2}$, et $\frac{3\,a}{2\,a}$, sive quod idem est, $\frac{3}{2}$, fractiones sunt, seu fractæ quantitates, quarum termini superiores sunt numeratores, inferiores verò, denominatores.

Quantitates fractas majores unitate, ex imperfectis partitionibus

majoris per minorem, et fractas quantitates unitate minores, ex partitione minoris per majorem, ortum suum ducere superius in divisione ostendimus.

> Ut divisis 9 per 2 producuntur $4\frac{1}{2}$, seu, si mavis, $\frac{9}{2}$ unitate majores. Item divisis 3 per 5 oriuntur $\frac{3}{5}$, ut in divisione superius ostendimus.

Unde omnis numerator vicem gerit quantitatis partiendæ; denominator verò, quantitatis partientis eam.

> Ut superiore exemplo $\frac{3\,ab}{2\,bc}$ idem significant quod $3\,ab$ divisa per $2\,bc$; sic $\frac{3\,a}{2\,a}$ idem valent quod $3\,a$ divisa per $2\,a$, seu brevius, 3 divisa per 2; sive tandem idem valet quod tres partes unitatis divisæ in duas; sic $\frac{3}{4}$ sunt tres quartæ unitatis, vel tres partitæ per quatuor, quod idem est.

Atque omnis quantitas, numeratorem et denominatorem habens, pro fracta habetur, et ut fractio computatur.

Hinc prudenter unitatem pro denominatore integris subjicimus, ut integræ cum fractis, quasi fractæ, computentur.

Facilius autem computantur fractæ, si earum termini contrahantur, et abbrevientur, priusquam inter operandum accreverint.

Abbreviantur, autem, et contrahuntur, partiendo terminos accretos per suum maximum communem divisorem.

Est autem maximus communis divisor, quo major dari nequeat perfecte dividens utrumque terminum.

Hîc habetur, partiendo terminum majorem per minorem, primò, et deinde semper partiendo præcedentem partitorem per suas reliquias, donec tandem nihil remanserit; et ultimus ille divisor (spretis quotis) est maximus communis partitor quæsitus.

Ut terminorum 55 et 15 maximus communis partitor sic habetur; partire 55 per 15, remanebunt 10; partire 15 per 10, remanebunt 5; partire 10 per 5 et nihil remanebit: 5 ergo sunt maximus communis partitor, partiens 15 per 3, et 55 per 11.

Verùm si ad unitatem partitorem perveneris, inabbreviabiles, discreti tamen sunt termini, aut se invicem habentes ut discreti.

Ut sint termini $5a$ et $3a$; partitis $5a$ per $3a$, remanent $2a$; inde partitis $3a$ per $2a$, remanet $1a$; per quod partitis $2a$, nihil remanet. Unde $5a$ et $3a$ non habent majorem partitorem unitate, seu $1a$; per quod si partiantur, habebunt se invicem ut discreti numeri 5 et 3, ut postea suo loco amplius dicetur.

Verùm hìc summopere cavendum est à partitione incommensurabilium quantitatum, cujus nullus in æternum erit finis, ut suo loco perspicuum evadet.

Ut denarii, et suæ bipartientis radicis, quam radicem quadratam vocant, nulla reperietur in æternum communis mensura; multò minus partitor ille maximus, ut suo loco.

Cum terminis inabbreviabilibus tanquam abbreviatis eorum defectu est operandum.

Habito maximo communi divisore, et per eum partito utroque termino, oriuntur novi termini abbreviati; et hæc operatio abbreviatio dicitur.

CAPUT VIII.

DE COMPUTATIONIBUS QUANTITATUM FRACTARUM.

Additiones et substractiones sunt fractionum ejusdem denominationis.

Si diversæ sunt denominationis, ad eandem reducantur.

Reducuntur autem, partitis utriusque denominatoribus per suum maximum communem partitorem, notatis quotis.

Inde multiplicando utrosque terminos prioris in quotum posterioris denominatoris, et fit nova prior; et utrosque terminos posterioris per quotum prioris denominatoris, et fit nova posterior ejusdem denominationis.

> Ut sint fractæ $\frac{2}{3}$ et $\frac{7}{9}$ ad eandem denominationem reducendæ; denominatorum 3 et 9 communis maximus partitor est 3, per quæ divisis denominatoribus, oriuntur 1 pro priore, et 3 pro posteriore; deinde, multiplicatis $\frac{2}{3}$, utroque termino, per posteriorem quotum 3, oriuntur $\frac{6}{9}$ pro priore novâ. Sic multiplicando $\frac{7}{9}$ per unitatem (prioris scilicet quotum), fiunt $\frac{7}{9}$ ejusdem denominationis cum $\frac{6}{9}$.

Harum jam ejusdem denominationis, addantur et substrahantur numeratores novi, retento novo et communi denominatore, et habebis additionis totam, et substractionis residuam.

> Ut superiore exemplo addantur novi numeratores 6 et 7, et oriuntur 13, quæ, cum denominatore communi 9, faciunt $\frac{13}{9}$ pro additionis totâ. Sic si substraxeris $\frac{6}{9}$ ex $\frac{7}{9}$, remanebit $\frac{1}{9}$ substractionis residua.

Multiplicantur etiam fractæ, partiendo singulas binas, quarum altera est numerator, altera denominator, in suum communem divisorem maximum, notatis omnium ultimis quotis, deinde multiplicando quotos numeratorum invicem, et fiunt novus numerator; et quotos denominatorum invicem, et fiunt novus denominator multipli quæsiti.

Ut sint invicem multiplicandæ $\frac{18}{20}$ et $\frac{35}{231}$; primò partiantur 18 et 20 per suum communem divisorem maximum 2, fient $\frac{9}{10}$ et $\frac{35}{231}$; inde partiantur 10 et 35 per 5, fient 2 et 7, hoc situ, $\frac{9}{2}$ et $\frac{7}{231}$; inde partiantur 9 et 231 per 3, fient 3 et 77, hoc situ, $\frac{3}{2}$ $\frac{7}{77}$; denique partiantur 7 et 77 per suum communem divisorem maximum 7, et fient 1 et 11, hoc situ, $\frac{3}{2}$ $\frac{1}{11}$: His peractis, duc hos numeratorum ultimos quotos 3 et 1 invicem, sic ultimos denominatorum 2 et 11, fientque illi 3, hi 22, hoc situ, $\frac{3}{22}$, multiplum quæsitum. Item sint multiplicandæ fractiones inabbreviabiles hæ $\frac{2a}{3}$ et $\frac{4}{5}$ invicem; multiplicentur primò numeratores $2a$ et 4 invicem, et fient $8a$, novus numerator; deinde denominatores 3 et 5 invicem, et fient 15, novus denominator; sunt ergo $\frac{8a}{15}$, multiplum quæsitum.

Hâc multiplicatione fractiones fractionum, imo, et fractiones fractionum iterum atque iterum fractarum, ad simplices fractiones reducuntur.

Ut duæ quintæ trium quartarum, sic notatæ, $\frac{2}{5}$ ex $\frac{3}{4}$, per præmissam fiunt primò $\frac{1}{5}$ $\frac{3}{2}$ per abbreviationem; inde, per numeratorum invicem ac denominatorum invicem multiplicationem, fiunt $\frac{3}{10}$, unica fractio simplex, idem valens quod superior fractio fractionum. Sic tres quartæ duarum tertiarum unius dimidii, sic notatæ, $\frac{3}{4}$ ex $\frac{2}{3}$ ex $\frac{1}{2}$, fient per abbreviationem, primò $\frac{1}{4}$ ex $\frac{2}{1}$ ex $\frac{1}{2}$, inde $\frac{1}{4}$ $\frac{1}{1}$ $\frac{1}{1}$ seu $\frac{1}{2}$ $\frac{1}{1}$ $\frac{1}{2}$; tandem ductis superioribus quotis invicem, et inferioribus invicem, fiunt $\frac{1}{4}$, idem valens quod $\frac{3}{4}$ ex $\frac{2}{3}$ ex $\frac{1}{2}$.

Partiuntur, autem, invertendo terminos divisoris, et inversos per partiendum multiplicando omnimodo ut superius in multiplicatione.

Ut sint $\frac{3}{10}$ penultimi exempli partiendæ per $\frac{3}{4}$; hujus divisoris inverte terminos, et fient $\frac{4}{3}$, quæ per $\frac{3}{10}$ multiplicatæ fient primò per abbreviationem $\frac{1}{10}$ $\frac{4}{1}$, deinde $\frac{1}{5}$ $\frac{2}{1}$, deinde per multiplicationem superiorum invicem, et inferiorum invicem, fient $\frac{2}{5}$, quotus optatus, et superioris multiplicationis examen.

Extractio autem fit extrahendo indicatam radicem tam ex numeratore quàm ex denominatore, sive ex quibuscunque terminis ejusdem rationis, et fient radicis quæsitæ termini.

Ut sit extrahenda radix bipartiens ex fractione $\frac{16}{25}$; extrahatur radix bipartiens 16, estque 4; inde radix bipartiens 25, estque 5; ex illo fit numerator 4, ex hoc denominator 5, hoc situ, $\frac{4}{5}$, radix bipartiens $\frac{16}{25}$ optata. Item sit extrahenda radix bipartiens ex $\frac{3}{4}$, seu, quod melius est, ex $\frac{48}{64}$; quanto enim majores sunt quantitates ejusdem rationis, tanto exactiores sunt radices, nisi perfecte extrahantur. Extrahe ergo radicem bipartientem è 48, quæ non habetur; ergo è 49, et ea erit 7; sic è 64 extrahatur eadem radix, estque 8; sunt ergo $\frac{7}{8}$ radix bipartiens $\frac{3}{4}$, veritati quàm proxima.

Radicales multiplicationes, et partitiones, et regulam proportionis, quia nihil aliud sunt quàm multiplicationes et partitiones repetitæ, ad præmissa referimus, ex quibus facile habentur.

Completis computationibus his, restituendæ sunt fractiones mixtæ, quarum scilicet numerator excedit denominatorem, ad suas integras et fractiones. Fit autem restitutio hæc, partiendo numeratorem per denominatorem, et emerget in quotiente integra quantitas, et reliquiæ erunt

numerator, et divisor erit denominator, fractioni illi mixtæ et adjunctæ.

Ut si ex computationibus completis provenerint $\frac{11}{4}$; dividantur 11 per 4, et fiunt 2 in quoto, et 3 supererunt; unde duæ integræ, et tres quartæ unitatis, hoc situ, $2\frac{3}{4}$, sunt idem quod superiores $\frac{11}{4}$, reformatæ et magis perspicuæ.

FINIS LIBRI PRIMI.

LIBER SECUNDUS.

DE LOGISTICA ARITHMETICA.

CAPUT I.

DE INTEGRORUM NOMINATIONE ET NOTATIONE.

Hactenus computationes quantitatum omnibus Logisticæ speciebus communium; sequuntur propriarum.

Tertiò, itaque, computationes vel sunt verinomiarum, vel fictinomiarum seu hypotheticarum quantitatum. Unde Logistica vel est verinomiarum, de quibus Lib. II. et III.; vel fictinomiarum seu algebraicarum, de quibus Lib. IV. agetur.

Verinomiæ sunt quantitates veris nominibus definitæ, quibus, quotæ sint multitudine, vel quantæ sint magnitudine, explicatur.

Verinomiæ aut sunt discretæ numero discreto, aut concretæ numero concreto nominatæ.

Unde, Logistica verinomiarum vel est discretarum quantitatum, quæ Arithmetica dicitur, de quâ hoc Lib. II.; vel concretarum, quæ Geometrica, de quâ Lib. III. agetur.

Arithmetica, ergo, est Logistica quantitatum discretarum per numeros discretos.

Numerus discretus est, quem suum unicum individuum numeratum metitur.

Numerus discretus aut est integer, aut fractus; unde Arithmetica est integrorum et fractorum.

Integri sunt, quos individua unitas numerata metitur.

Vocales integrorum nominationes quodque suppeditat idioma; ut Latinum,—unum, duo, tria, quatuor, &c.

Scripta autem integrorum nomina, seu notæ, novem sunt significativæ hæ: 1 unum, 2 duo, 3 tria, 4 quatuor, 5 quinque, 6 sex, 7 septem, 8 octo, 9 novem.

Hæ, diversis locis, diversos significant numeros.

Præter has novem notas, seu figuras, est circulus 0, qui nullibi locorum quicquam significat, sed locis vacuis supplendis destinatur.

Locorum series à dextrâ in sinistram consideratur, in quorum primo, figura suo jam dicto valore nominatur; secundo, decuplo; tertio, centuplo; quarto, millecuplo; quinto, decies millecuplo; sexto, centies millecuplo; septimo loco, millies millecuplo; octavo, decies millies millecuplo valore nominatur. Et ita deinceps in infinitum, per decuplum incrementum semper progrediendo.

> Ut 7 sunt septem; at 70 sunt septuaginta; 700, verò, sunt septingenta: Sic 8000, octo millia; 60000, autem, sexaginta millia. Unde 68777, sexaginta octo millia, septingenta septuaginta septem, indicant. Item 90630, nonaginta millia, sexcenta triginta, significant: Et ita de aliis.

Hinc fit majorum numerorum facilis nominatio, si, post tertiam quamque figuram constituto puncto, primum punctum millia appelles; secun-

dum, millia millium; tertium, millena millia millium; quartum, millies millena millia millium; et ita de reliquis punctis. Figuræ, verò, primo loco à punctis constitutæ, suo valore nominentur; secundo, verò, à punctis loco, decuplo valore; tertio denique, centuplo sui valore nominentur.

Ut hic numerus, 4734986205048205, sic pungatur, 4. 734. 986. 205. 048. 205. Et hæc erit ejus nominatio: quatuor millies mille millena millia millium, septingenta triginta quatuor millies millena millia millium, nongenta octoginta sex millena millia millium, ducenta quinque millia millium, quadraginta octo millia, ducenta et quinque: Et ita de aliis.

CAPUT II.

DE ADDITIONE ET SUBSTRACTIONE INTEGRORUM.

Hactenus integrorum nominatio et notatio; sequitur computatio; et primò de additione et substractione.

In additione spectatur situs, et operatio seu praxis.

Situs est, ut numeri numeris subscribantur; ita ut, à dextris incipiendo, figuræ primæ primis, secundæ secundis, et reliquæ reliquis directe substituantur, ductâ sub numero infimo lineâ.

Operationis tria sunt præcepta.

Primum, ut primi loci figuræ, omnes in unam summam colligantur, et, hujus summæ, prima tantum figura eis, infra lineam, subscribatur; cæteris, si quæ sint, animo reconditis.

Secundum, ut cæteræ hæ, animo reconditæ, una cum omnibus figuris sequentis loci, in unam summam colligantur, et, hujus etiam summæ, prima tantum figura eis, infra lineam, subscribatur; cæteris ejus summæ

figuris, si quæ sint, animo reconditis: Et hæc operatio in ultimam omnium figuram repetenda est.

Tertium, ut ultimi loci figuræ, cum novissime animo reconditis, (si quæ sint) in unam summam collectæ, loca sinistima compleant.

Ut quintum caput Genesecos exhibet annos à creatione Adami et mundi 130 ad Sheth; hinc 105 ad Enosch, hinc ad Kenan 90, hinc ad Mehalalelem 70, ad Jered 65 annos, hinc ad Henoch 162, inde ad Methuselach 65, ad Lamech 187, hinc ad natum Noah 182, à nato Noah ad initium diluvii 600. Quæritur, his jam additis, summa annorum à condito orbe ad diluvium? Anni ergo omnes recto situ, ut à margine, constituantur; inde, primò, dextimæ seu primi loci figuræ 5, 5, 2, 5, 7, 2, addantur, et fient 26; subscribantur 6 directe, 2 autem in mente reserventur; secundò, hæc 2 una cum secundi loci figuris 3, 9, 7, 6, 6, 6, 8, 8, adde, fient 55, quorum priorem notam 5 scribe, posteriorem mente reserva; tertiò, hunc reservatum quinarium, una cum 1, 1, 1, 1, 1, 6, ultimi loci figuris, adde, fient 16, quæ locis sinistimis ponantur, fitque tota summa 1656.

130
105
90
70
65
162
65
187
182
600
————
1656

Additio, tamen, proprie est duorum numerorum, ut tertius inde producatur.

Ut sint addendi 9754862 atque 863556; hi, ex superioribus præceptis, producunt totum 10618418, situ quo à margine. Hujus examen habes in substractione sequente.

9754862
863556
————
10618418

In substractione est situs, et operatio.

Situs, ut in additione, à dextris incipit; substrahendi autem et minuendi minor infimo loco, major medio loco, et residuus summo

constituitur, lineâ inter summum et medium interpositâ; ita ut à dextrâ figuræ primæ primis, secundæ secundis, et reliquæ reliquis, directe superscribantur.

Operatio, contra, à sinistris aptissime inchoatur, aut si mavis à dextris, hoc utcunque servato unico præcepto:

Ut scilicet figura quæque inferior, aut nihil ubi nihil subest, ex superpositâ illi figurâ illâ non minore, aut minore additis 10, substrahatur; et residua integra illi superscribatur, si modo inferioris summa, hinc dextrorsum extensa, non excedit summam ei superscriptam; alioquin, si excedit, tunc præfata residua figura minuta unitate, si sit numerus, vel aucta novenario, si sit 0, illi superscribatur.

2690997393
2738154098
47156705

Ut sint 47156705 substrahenda ex 2738154098 numero minuendo: Sit situs hujus supra, illius infra, ut in margine, et dextimæ dextimis figuræ respondeant: Proinde primò à sinistimis 27 aufer nihil (nihil enim subest), remanent 27 minuenda unitate, quia 4, &c. subscripta excedunt 3, &c. superscripta; et ita 26, residua, sunt superscribenda prioribus 27; deinde aufer 4 ex 3, additis tamen prius his 10, ut 4 inde auferri possint, et supersunt 9, quæ absque minutione sunt superscribenda tribus, quia 7 minora sunt quàm 8; deinde aufer 7 ex 8, remanet unitas unitate minuenda, et ita 0 est superscribendum, quia dextrorsum 156, &c. exuperant eis superposita 154, &c.; deinde auferatur 1 ab 1, remanet 0; sed quia sequentia 56, &c. exuperant superposita 54, &c., ideo pro 0, 9 sunt superscribenda; similiter aufer 5 ex 5, superest 0, sunt ergo 9 superscribenda, quia 6 exuperant 4 superscripta; inde 6 ex 14 aufer, manent 8, scribe tantum 7, quia subjecta figura 7 excedit 0 superpositum ei; deinde aufer 7 à 10, remanent 3, integre superscribenda, quia subjectum 0 minus est quàm 9 superscripta; deinde aufer 0 ex 9, restant 9, integre superscribenda,

quia 5 inferius minora sunt 8 suprapositis; tandem aufer 5 ex 8, et supersunt 3, integre superscribenda, quia nihil post extimam figuram subjectum sequitur quod exuperat superpositum: Et ita habes residuum quæsitum, 2690997393.

Hujus operis examen fit per,—addendo 47156705 ad 2690997393, et restituentur 2738154098; vel per,—substrahendo 2690997393 ex 2738154098, et restituentur 47156705: Et ita in aliis.

Aliud exemplum, et superioris additionis examen.

Sint substrahenda 863556 ex 10618418, residuus numerus erit 9754862, ut superius in additione. Idem proveniet si à dextris sinistrorsum operatus fueris.

Si offeratur major numerus ex minore substrahendus, minorem nihilominus semper ex majore, per præmissam, substrahes; verùm, residuum minutionis copulâ prænotabis, et inde numerus defectivus orietur.

Ut sint substrahenda 10618418 ex 863556, fiat, per præmissam, substractio horum ex illis, et provenient (ut in præcedente exemplo) 9754862, quæ, conversa in —9754862 defectivum, sunt residuus numerus quæsitus; ut de quantitatibus defectivis, in genere, superius diximus, Lib. I. cap. 6.

CAPUT III.

DE MULTIPLICATIONE INTEGRORUM.

Multiplicatio aut est per unicam, aut plures figuras.

Multiplicatio per unicam, aut est unius aut plurium.

Multiplicatio unius per unicam, seu singularum per singulas, memoriter, ex subjectâ tabulâ, discenda est quàm promptissime.

	9	8	7	6	5	4	3	2	1
1	9	8	7	6	5	4	3	2	1
2	18	16	14	12	10	8	6	4	
3	27	24	21	18	15	12	9		
4	36	32	28	24	20	16			
5	45	40	35	30	25				
6	54	48	42	36					
7	63	56	49						
8	72	64							
9	81								

Ut si, quid ex 7 et 8 invicem multiplicatis oriatur, interrogatur ? Quære majorem 8, scilicet in supremâ, et minorem 7 in sinistimâ lineâ, et in angulo communi offendes 56, multiplum optatum.

Oblitus multipli producti ex duabus majoribus quibuscunque figuris, earum à denario defectus invicem multiplica, provenietque figura dextimi loci ; deinde minorem defectum à minore figurâ, aut majorem à majore, auferto, remanebit figura sinistima, complentes quæsitum.

$$\begin{array}{cc} 8 & 2 \\ 7 & 3 \\ \hline 5 & 6 \end{array}$$

Ut si oblitus sis quantum septies 8, vel (quod idem est) octies 7, effecerint, earum à denario defectus 3 et 2 invicem multiplica, et provenient 6, figura dextima ; deinde aufer 2 à 7, vel 3 ab 8, et restabunt 5, figura sinistima. Itaque 56 sunt quæsitum multiplum ex 7 et 8.

Multiplicationis plurium per unicam tam situs quàm operatio à dextris orditur, procedendo ordine lævorsum ; ac multiplicando, etiam, sive supra sive ante multiplicantem constituto, sub utroque ducitur linea. Multiplicatio plurium per unicam tria habet in operatione præcepta.

Primum est, ut dextima figura per datam unicam multiplicetur, et multipli unica figura, vel dextra (si duæ sint), subnotetur ; sinistra, autem, si quæ sit, animo reservetur.

Secundum est, ut figura hæc, animo reservata (si quæ sit), addatur ad

multiplum sequentis figuræ, in unicam multiplicatæ; aggregati, verò, unica figura, vel dexterior (si duæ sint), subnotetur, sinisterior autem (si quæ sit) animo reservetur, addenda ut prius; et hæc operatio in sinistimam seu ultimam figuram est repetenda.

Tertium est, ut ultimæ figuræ multiplum, cum novissime animo reconditâ (si quæ sit), in unicam summam collectum, sinistimis locis integrum subnotetur.

Ut sint 865091372 quintuplicanda, seu per 5 multiplicanda: Eorum situs, utcunque sit, ut in margine: Primò, itaque, figura dextima 2, in unicam multiplicatricem 5, multiplicetur; fient 10 (duæ figuræ), quorum dextram 0 subnoto, 1 autem reservo; deinde duco, seu multiplico, 7 in 5, fient 35; quibus addo 1 reservatum, fiunt 36; quarum dextram 6 subnoto, tribus mente reconditis: Pergo;—ter 5, seu quinquies 3, sunt 15, et 3, mente reservata, fiunt 18, subnoto 8, et 1 reservo; sequitur quinquies 1, fiunt 5, et 1 reservatum fiunt 6, unicâ figurâ subnotandâ, nihilo mente reservato; deinde, quinquies 9 sunt 45; subscribo 5, mente autem 4 reservo; sequitur quinquies 0, quod quidem nihil est; hoc, cum 4 reservatis, facit 4 subnotanda: Prosequor;—quinquies 5 sunt 25; noto 5, reservo 2; inde duco 6 in 5, proveniunt 30, et cum reservatis 2 fiunt 32; scribo 2, reservo 3; tandem, duco 8 in 5, fiunt 40, quibus addo 3 reservata, jam omnium ultimò, fiuntque 43, quæ quidem integre, in locis omnium sinistimis, scribo. Totum, itaque, multiplum quæsitum est 432545860.

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 865091372
         5
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4325456860

 865091372(5
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4325456860
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Superest, jam, plurium figurarum per plures multiplicatio.

Multiplicatio plurium figurarum per plures, præter præmissa, tria habet præcepta.

Primum, ut totum multiplicandum (per jam præmissa) multiplicetur in quamque figuram multiplicantis, sive à dextris, sive sinistris incipere libuerit.

Secundum, ut cujusque figuræ multiplum habeat suam dextimam figuram directe notatam sub figurâ multiplicante, et cæteras ordine sinistrorsum sequentes.

Tertium præceptum est, ut sub his particularibus multiplis ducatur alia linea, atque omnia multipla in unum aggregatum addantur, quod quidem erit totale multiplum, et quæsitum totius multiplicationis productum.

Ut sit præcedens numerus 865091372 multiplicandus in 92105: Collocentur, utrolibet situ, ut à margine; deinde à quovis termino multiplicantis incipe, sive dextro sive sinistro ordine; exempli gratiâ, à dextro: Duc itaque totum multiplicandum per 5; fient, per præmissa, 4325456860, quæ ita locentur ut figura dextima 0 sub figurâ multiplicante 5 directe statuatur, cæteris sinistrorsum sequentibus: deinde, idem multiplicandum per 0, quod nihil est, multiplicetur, et nihil proveniet; nihil, ergo, hujus multiplicationis, notari opus habet: sequitur multiplicandum per 1 ducere; proveniet idem ex eodem, viz. 865091372, quarum figurarum dextima 2 sub suâ multiplicante figurâ 1 collocetur, cæteris sinistrorsum sequentibus: deinde idem multiplicandum duplicetur, seu per 2 ducatur, et fient 1730182744, quarum situs incipiat sub suo multiplicante 2, et inde sinistrorsum progrediendo: deinde totum multiplicandum duc in 9; fient 7785822348, quæ, sub 9 situs initium habentes, hinc ordine lævorsum notentur: Tandem, omnia invicem particularia multipla, lineis interclusa, adde, eo quem habent situ, et fient 79679240818060, pro completo et quæsito multi-

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    865091372
        92105
   ----------
   4325456860
  865091372
 1730182744
7785822348
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79679240818060
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865091372(92105
7785822348
 1730182744
   865091372
    4325456860
79679240818060
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plicationis multiplo. Nec secus eveniet ex secundo situ, incipiendo, nimirum, multiplicationem per sinistimam figuram multiplicantis 9, ut ex secundo schemate patet. Similiter etiam in similibus.

Hujus examen habes in partitione sequente, ut et illius hoc est examen.

Multiplicatio trium, aut plurium numerorum, perficitur multiplicando primum in secundum, et horum multiplum in tertium, et horum rursus multiplum in quartum ; et ita in ultimum.

Ut sint multiplicanda invicem 5, 4, 2, et 3 ; duc 5 in 4, fient 20 ; secundò, duc 20 in 2, fient 40 ; tertiò, duc 40 in 3, fient 120, omnium multiplum.

CAPUT IV.

DE PARTITIONE INTEGRORUM.

Partitio minoris per majorem non fit aliter quam interponendo lineam, inter superpositum partiendum et infrapositum partitorem, et totus hic quotus fractio est, unitate minor.

Ut sint 3 partienda per 5, sit quotus $\frac{3}{5}$, quæ pronunciantur, tres quintæ unitatis, vel, tria divisa per quinque ; et fractio est.

Unde hinc ortum habent numeri fracti, ut de quantitatibus fractis in genere, cap. 7, docuimus. De his tractabit secunda Arithmeticæ pars.

Partitionem æqualis per æqualem, in omnibus quantitatibus, unitatem producere superius diximus.

In partitione, tandem, majoris numeri per minorem, producitur quotus semper unitate major ; et proprie huc spectat hæc partitio.

In hac majoris per minorem partitione, spectatur situs et operatio, quorum uterque incipit à lævâ, dextrorsum tendens. Situs aptissimus est, ut sinistima partitoris figura sub sinistimâ partiendi eâ minore, aut ante sinistimam partiendi eâ majorem, constituatur, cæteris figuris ordine dextrorsum sequentibus, post quarum dextimam immediate incipiat parenthesis, quoto capiendo destinata ; et sub omnibus ducatur linea.

Operationis quatuor sunt præcepta.

Primum, ut diligenter exquiratur, quoties partitor subduci potest ab eis supremarum figurarum sinistimis, quæ supra vacuum quoti locum, parenthesi proximum, terminantur : sumptâ, plerumque, hujus quoti conjecturâ, ex primis ad primas relatis ; deinde, ut figura hunc quotum notans, proximo, post partitorem et parenthesin, vacuo loco constituatur.

Secundum, ut totus partitor, in hanc figuram quoti recens acquisitam, multiplicetur, et multiplum suo debito loco, per cap. 3, constituatur.

Tertium, ut hoc recens multiplum, ex supremis figuris ei directe superpositis, substrahatur, atque deletis etiam figuris, et substractis et ex quibus substrahuntur, supernotentur residuæ, per cap. 2 hujus.

Quartum, ut repetantur hæ tres operationes, usque quo totum partiendum deleatur, et residua, seu reliquiæ, aut nullæ aut partitore minores prodeant, dumque etiam nulla quoti loca, usque in dextimam partiendi figuram, vacua relinquantur ; atque tandem, post quotum dextrorsum ducatur linea, cui superpositæ reliquiæ, et suppositus divisor, quoti fragmenta unitate denotabunt.

Exempli gratiâ : Sint partiendi dies anni bissextilis 366, per dies hebdomadis 7, ut sciatur quot sint in anno bissextili heb-

366
7(5

domades: Statuantur ut à margine; deinde primò inquirendum est, quoties auferri possunt 7 ex 36 (etenim 36 sunt sinistimæ illæ figuræ superpositæ, quarum terminus, videlicet, figura 6, desinet supra locum vacuum quoti, proximum parenthesi), et invenies quinquies 7 in 36; statue, ergo, 5 in primo quoti loco vacuo; secundò, multiplica 7 per 5, fient 35 debito loco reponenda, (videlicet 5 sub 5, et 3 sinistrorsum); tertiò, aufer 35 ex 3 et 6 superpositis, et, eis deletis, relinquitur unitas superius notanda.

0
1
366
7(52$\frac{2}{7}$
35
14

Sequuntur ergo reliquiæ 16, ad et supra sequentem locum vacuum quoti, ex quibus 16, (repetendo opus ut prius), recole quoties 7 auferri possint, et hîc bis fieri posse, relictis 2, conspicies; sequente, ergo, et ultimo quoti loco, notentur 2; deinde ducantur 2 per 7, fiunt 14, quæ rite collocata cadunt sub 16, ex quibus aufer 14, et, utrisque inde deletis, relinquentur 2 dies, seu $\frac{2}{7}$ hebdomadis, quoto annectendæ, ut fiat verus quotus, 52 hebdomadarum et $\frac{2}{7}$, duarum scilicet septimarum hebdomadis, in anno bissextili.

Aliud exemplum.

Sint 861094 partienda per 432: constituta ut à margine; videlicet, 4 ante 8, quia 4 minora sunt quàm 8; deinde con-

118
141
402
429
861094
432(1993$\frac{118}{432}$
432
3888
3888
1296

sideratur quoties 432 è 861 auferri possint, conjecturam ex 4 et 8 sumendo, vel ex 43 et 86, et invenies 4 bis ab 8, et 3 bis à 6 auferri posse; sed tamen 2 non posse bis ab 1 auferri; unde (fallente hâc conjecturâ) pones omnes ab omnibus semel tantum auferri posse; scribatur ergo 1, pro primâ quoti figurâ, per quod multiplica 432, et fient 432,

ex 861 auferenda, et restabunt 429. Itaque, à 4290 perscrutare quoties poteris auferre 432, sumptâ conjecturâ ex 4, quæ in 42 novies habentur, et remanent 6; 69 etiam novies continent 3, et satis multa præterea remanebunt, quibus novies contineantur 2; ideo 9 sequente loco vacuo, pro secundâ figurâ quoti, ponitur; deinde, ductis 432 per 9, fient 3888, substrahenda ex 4290, et remanent 402. Itaque, perscrutare quoties à 4029 auferri possint 432, simili quâ prius conjecturâ, et invenies rursus novies hoc etiam fieri posse; per alia ergo 9, quoto adjecta, multiplica 432, et fient, ut supra, 3888 substrahenda ex 4029, et supersunt 141. Considera, ergo, ex 1414 quoties 432 auferri possint, et id ter fieri posse reperies.

Ultimo ergo quotientis loco 3 ponantur, per quæ multiplica 432, fient 1296, auferenda ex 1414 suprapositis, et tandem supersunt reliquiæ 118, sive $\frac{118}{432}$, quæ quoto adjiciantur, et perfectum quotum $1993\frac{118}{432}$ reddent.

Tertium exemplum, et superioris multiplicationis examen.

Sint 79679240818060 partienda per 865091372; constituantur ut à margine; deinde considera quoties 8650 auferri possint ex 79679, seu quoties 8 ex 79, et perspicies novies id fieri posse, relictis etiam quot sufficiunt pro 6, et 5, et reliquis sequentibus, etiam novies auferendis; deinde per 9, loco quoti posita, multiplica partitorem 865091372, et fient 7785822348, debito loco subnotanda, et ex illis superpositis auferenda, et remanebunt 1821017338 partienda per oblatum partitorem 86509, &c., et percipies hæc ex illis (nimirum 8 ex 18, et cætera ex cæteris)

```
                     0
     432545368
    908345394
  182101733
7967924081 8060
865091372(92105
---------------
7785822348
 1730182744
   865091372
     4325456860
```

bis tantum auferri posse; per 2, ergo, quoto apposita, multiplica partitorem, et fient 1730182744, ex superioribus reliquiis auferenda, et relinquentur 908345940, ex quibus, datus ille partitor semel tantum auferri potest; per 1, ergo, quoto appositum, multiplica partitorem, et exsurget ipse idem partitor 865091372; quo ex 908345940 subducto, relinquuntur 432545686, ex quibus partitor 865091372 ne semel quidem auferri potest; ideo, posito 0 in quoto, perges, et quoties partitor ille ex 4325456860 auferri possit, quæras, et invenies id quinquies fieri (factâ conjecturâ per 8, quæ ex 43 quinquies auferri possunt); per 5, ergo, quoto adjecta, multiplica partitorem, et producuntur 4325456860, ex superioribus etiam his æqualibus auferenda, et nihil superest reliquium: Perfecta est ergo hæc partitio, et 92105 sunt quotus integer; atque approbat penultimum exemplum multiplicationis, et ab eo approbatur. Ita est de aliis.

CAPUT V.

DE MULTIPLICATIONIS ET PARTITIONIS COMPENDIIS MISCELLANEIS.

Multiplicationes per 10, 100, 1000, aut per alia ex unitate et circulis quotvis conflata, facile fiunt adjiciendo, solum, tot circulos multiplicando à dextra, quot habet multiplicator.

Ut sint 865091372 decuplicanda, fientque 8650913720; aut centuplicanda, fientque 86509137200; aut per 10000000 multiplicanda, et provenient 8650913720000000.

Contra, partitio 10, 100, 1000, aut alia ex unitate et circulis quotvis conflata, facile fit abscindendo tot, ex dextimis figuris partiendi, quot

circulos habet partitor : abscissæ tamen figuræ supra lineam, et partitor infra lineam, sunt constituendæ, et quoto adjiciendæ ad quoti fragmenta notandum.

Ut sint 865091372 partienda per 100, fient $8650913\frac{72}{100}$ pro quoto. Item sint 865091372000089 partienda per 10000000, fit quotus $865091372\frac{89}{100000000}$.

Hinc sequitur, multiplicationes et partitiones per quoscunque numeros auctos circulis facile fieri, spretis primò in multiplicatione circulis, donec multiplicandi et multiplicantis reliquæ figuræ invicem ducantur ; inde multiplo tot circulos restituendo ; atque pro partitione abscindendo et circulos à partitore et tot figuras à partiendo ; atque reliquum partiendi per reliquum partitoris dividendo, et reliquias cum abscissis, positâ inter eas lineâ, adjiciendo.

Ut sint multiplicanda 65294 per 2300, fient primò 1501762 (ex ductu 65294 in 23), deinde adjiciendo 00 fient 150176200, pro toto multiplo. Item sint partienda 65294 per 2300, constituantur ut à margine, ita ut pro 00 abscindantur gnomone 94, hoc modo |94 ; deinde partiantur 652 per 23, et fiet quotus, cum fragmentis residuis, $28\frac{894}{2300}$, seu (quod idem est) $28\frac{447}{1150}$.

```
  1̸0̸8
  6̸5̸2̸|94
23(28
  4̸6̸
  1̸8̸4̸
```

Primo intuitui facilem se objicit bipartitio.

Ut sint bipartienda 65294 ; quis non primo intuitu conspicet 3 esse dimidium 6, et 2 quinarii, relicto 1, constituente cum sequente binario 12, quorum dimidium est 6 ; deinde sequuntur 9, quorum dimidium sunt 4, relicto 1 (denario scilicet); sequuntur ergo omnium ultimo 14, quorum dimidium est 7 ; ex quibus omnibus 3, 2, 6, 4, 7, fiunt 32647, dimidium numeri oblati, dicto citius repertum, ut à margine notatur.

65294
32647

Hinc sequitur facillima quintuplicatio, nimirum, decuplum bipartiendo.

Ut sint 865091372 quintuplicanda; ejus decuplum (per primam sectionem hujus) est 8650913720, cujus dimidium 4325456860 est quintuplum oblati facillime inventum.

Et contra, facilis est quinquepartitio, duplum decupartiendo.

Ut sint 4325456860 quinquepartienda ; primò duplentur, fientque 8650913720 ; deinde (ex sectione 2 hujus) partiantur per 10, fientque 865091372, quinta pars oblati quæsita.

Nec difficilis erit nonuplatio, numerum à suo decuplo auferendo ; aut quadruplatio, numerum à suo quintuplo abstrahendo ; aut sextuplatio, numerum ad suum quintuplum adjiciendo ; aut ipsa duplatio, numerum ad sibi æqualem addendo. Siquidem hæ multiplicationes, facillimâ additione, aut substractione, perficiuntur.

Atque ex his rursus, simili facilitate, habetur triplicatio, septuplicatio, et octuplicatio, atque ita per omnes novem figuras multiplicatio.

Exemplorum loco, sint præcedentia et sequentia.

Ubi autem multiplicator omnes novem figuras, aut earum potissimam partem, complexus fuerit, vel partitoris quotus prolixus fore videtur, præstat singularium multipla, sive per præmissa compendia, sive per additionem continuatam, sive, omnium facillime, per ossa Rhabdologiæ nostræ, imprompt u habere, atque, ex eis sic acquisitis, integram multiplicationem aut partitionem contexere.

Ut, pro examine ultimi exempli cap. de multiplicatione, videamus quid 92105 multiplicata per 865091372 producant: Duc ergo 92105 per 2, 3, 4, 5, 6, 7, 8, 9, sive per præmissa compendia, sive per continuam additionem simpli ad simplum, ad

Similia.		
	092105	Simplum
	184210	Duplum
	276315	Triplum
	368420	4m
	460525	5m
	552630	6m
	644735	7m
	736840	8m
	828945	9m
	921050	Decuplum, examinis gratiâ.

```
865091372
736840|276315
 552630|644735
  460525|184210
   828945
    092105
79679240818060
```

duplum, triplum, quadruplum, et ad cætera sub decuplo multipla; et fient ordine multipla ut à margine. Ita tamen, ut singula multipla eodem figurarum numero constent, præponendo deficientibus 0 sinistrorsum; deinde, pro multiplicationis compendio, hæc singularia multipla eo ordine collocentur, quem multiplicans 865091372 indicat, nimirum, sub 8 incipiat octuplum è tabellâ extractum, sub 6 sextuplum, sub 5 quintuplum, et ita de cæteris ut subsequitur; quibus tandem omnibus additis, proveniunt 79679240818060, multiplum quæsitum, et illi consonum quod supra repertum est.

Similiter pro examine superiorum, tam multiplicationis quàm partitionis, sint 79679240818060 partienda per 92105: Constituantur situ quo à margine (per cap. 4 hujus); deinde ex 796792, numerum tabulæ proxime minorem, viz. 736840, aufer, et figuram quoti ei in tabulâ respondentem, 8 viz., pro quoto pone, et supererunt 599524; ex quibus aufer sextuplum partitoris, numerum scilicet tabulæ illi proxime minorem, qui est 552630, et 6 adjiciantur quoto, relictis 468940; ex quibus aufer 460525, quintuplum partitoris, et fiet quotus jam 865, relictis 84158, ex quibus ne simplex partitor 092105 auferri vel semel possit. Ideo erit jam quotus 8650, et reliquiæ jam 841581 supererunt; ex

```
         66315
        34263
       12636
      84158
     46894      0
    59952  18421
  79679240818060
92105)865091372
  736840  644735
   552630  184210
    460525
      828945
       092105
        276315
```

quibus aufer 828945, partitoris noncuplum tabulatum, et fiet jam quotus 86509, relictis 126368 ; ex quibus aufer 092105, unicum scilicet et simplum partitorem ; et concreto jam quoto 865091, supersunt 342630 ; ex quibus 276315, triplum scilicet partitoris, aufer, et erit jam quotus 8650913, et supererunt 663156 ; ex quibus aufer 644735, et ex quoto jam 86509137 supersunt 184210 ; ex quibus aufer partitoris duplum tabulatum, et nihil supererit, producto toto et integro quoto 865091372 superioribus conveniente.

Hactenus multiplicatio et partitio simplex, cum suis compendiis : sequuntur multiplicatio et partitio radicalis.

CAPUT VI.

DE RADICALI MULTIPLICATIONE ET PARTITIONE INTEGRORUM.

Multiplicationis praxis ex ipsâ ejusdem definitione patet. Si enim unitatem in radicem multiplicaveris, fit ipsa radix ; quam si secundò in radicem duxeris, fit duplicatum ; quod si tertiò duxeris duplicatum in radicem, fit triplicatum ; quod, quidem, si quartò in radicem duxeris, oritur inde quadruplicatum ; atque sic deinceps quintuplicatum, sextuplicatum, etc., secundum qualitatem indicis.

Ut sint 235 multiplicanda ad indicem 4, seu quadruplicanda : Primò multiplica unitatem per 235 radicem, et fiunt 235 ; quæ secundò duc in radicem 235, fientque 55225, oblatæ radicis duplicatum ; quod et per radicem tertiò multiplicato, et fient 12977875, oblatæ triplicatum ; quod adhuc per radicem 235 quartâ vice multiplicato, et fient 3049800625, optatum oblatæ radicis quadruplicatum.

Unde, radicem aliquoties ab unitate ducere, idem est quod tot æquales radici invicem ducere.

Ut superiore exemplo, radix 235 quatuor multiplicationibus simplicibus ab unitate ducitur, et fit quadruplicatum 3049800625; et si 235, 235, 235, et 235, invicem duxeris, idem emerget radicatum, seu quadruplicatum; et ita de aliis.

Radices quarum indices sunt compositi, facilius per componentes multiplicari et extrahi posse, quàm per compositos indices, superius diximus.

Ut superiore exemplo, facilius est radicaliter duplicare 235, fientque 55225; atque hæc rursus duplicare, seu in 55225 ducere, fientque 3049800625, superius quadruplicatum: nam quadruplicare idem est quod duplicatum duplicare. Item sint 10 radicaliter sextuplicanda; ea fient continuatâ ab unitate multiplicatione,—primò 10; secundò 100, duplicatum; tertiò triplicatum, 1000; quartò quadruplicatum, 10000; quintò quintuplicatum, 100000; sextò, tandem, sextuplicatum, 1000000. Verùm, per dictum compendium, si radicaliter triplicaveris duplicatum 100, seu duplicaveris triplicatum 1000, proveniet inde sextuplicatum 1000000; quàm antea aliquantum expeditius. Idem enim est sextuplicare radicaliter quod triplicare duplicatum, aut duplicare triplicatum radicis. Bis enim tria, aut ter duo, idem sunt quod sex.

Est et alia radicalis duplicationis praxis, quàm per continuatam multiplicationem; nimirum, radicem multiplicandam in duas partes secando, et duplum multipli, quod fit ex ductu prioris in posteriorem, una cum duarum partium duplicatis, addita faciunt duplicatum quæsitum.

Ut si radix oblata 35 radicaliter duplicanda, secentur in partes, priorem 30, posteriorem 5; duc 30 in 5, fient 150, quorum duplum est 300; quibus adde duplicatum 30 et duplicatum 5, quæ sunt 900, et 25, fientque 1225, duplicatum totius numeri 35. Nec secus eveniet ex ductu 35 in 35. Item si prorogetur radix, sit 352; sint partes, prior 350, posterior 2; duc invicem, fient 700, quorum duplum est 1400; quibus adde duplicatum 2, quod est 4, et duplicatum 350 superius inventum, quod est 122500; et ex his tribus summis, 122500, 1400, et 4, additis, fiunt 123904, duplicatum quæsitum radicis 352. Experire multiplicando 352 in 352, et idem invenies. Eodem modo operandum foret in quartam usque figuram, foretque 3521. Ex anterioris partis 3520 duplicato jam habito 12390400, et posteriore parte 1, habebitur 12397441, duplicatum totius numeri, seu radicis, 3521.

Est etiam et alia radicalis triplicationis praxis; nimirum, radicem in duas partes secando, et triplum partis cujusque in alterius duplicatum ducendo, et singulas partes triplicando; ex his enim quatuor summis additis, producitur radicis oblatæ triplicatum.

Ut sit radix oblata 35 radicaliter triplicanda, secentur ut supra in partes 30 et 5; duc triplum 5, quod est 15, in duplicatum 30, quod est 900; et triplum 30, quod est 90, in duplicatum 5, quod est 25; et fiunt illa 13500, hæc verò 2250; deinde radicaliter triplicentur 30, et fiunt 27000; radicaliter etiam triplicentur 5, et fient 125; ex his ergo quatuor summis, 13500, 2250, 27000, et 125, additis, fiunt 42875, oblatæ radicis triplicatum quæsitum. Experire per multiplicationem continuatam 35, 35, et 35, invicem, et in idem incides. Similiter si hìc prorogetur radix, sitque 351; partiantur in anteriorem partem 350, et posteriorem 1, ex quibus, modo præfato, habebis has quatuor summas, 42875000,

367500, 1050, et 1 ; quibus additis, fient 43243551, triplicatum quæsitum radicis oblatæ 351.

Sunt et quadruplicationis, quintuplicationis, sextuplicationis, et aliarum radicalium multiplicationum, particulares praxes ad inveniendum radicatum indicis cujusque. Sed quia tum modus per continuatam multiplicationem primâ sectione descriptus generalis est, et satis facilis, tum ex converso regularum extractionum haberi possunt omnes hæ particulares praxes, has ideo hoc loco prætermittimus.

Partitionis etiam praxis ex ipsâ ejusdem definitione patet. Si enim radicatum per radicem in unitatem usque partitus fueris, numerus partitionum est radicalis quotus, seu index quæsitus.

Ut sint 55225 radicatum dividendum per radicem 235, emerget primus quotus 235, quibus per 235 divisis, emerget 1, secundus et ultimus quotus. Est ergo binarius hujus partitionis index, et radix est bipartiens radicati. Item si radicatum 12977875 per radicem 235 partiendum sit, emergit, primâ partitione, 55225, secundâ, 235, tertiâ, 1. Unde partitio hæc tripartitio est, et index est ternarius, et radix tripartiens dicitur.

Numerus indicis, seu qualitas radicis, habetur, tam descendendo à radicato ad unitatem, per partitionem, quàm ascendendo ab unitate ad radicatum, per multiplicationem : In utroque enim, numerus operationum est index, et radicis qualitas.

Ut superiore exemplo, sicut radicatum 12977875, partitum per 235 radicem, tertiâ partitione exhibet unitatem ; ita, ab unitate continuata multiplicatio 235 radicis, tertiâ multiplicatione in idem radicatum incidet ; unde utroque modo arguitur ternarius esse index.

Hinc fit, quod partitionis radicalis rarus sit usus in computationibus,

cum suo munere etiam fungatur multiplicatio. In his autem, multiplicatio radicalis, quæ ab unitate in radicatum, aut partitio, quæ à radicato in unitatem præcise non incidit, inutiles sunt. Arguunt enim oblatam radicem non esse perfectam radicem radicati.

CAPUT VII.

DE INVENIENDIS REGULIS EXTRACTIONUM RADICALIUM.

Uniuscujusque radicis propria est, et particularis, extrahendi regula.

Regula quæque extractionis consistit in resolutione radicati in sua supplementa.

Supplementum est, differentia duorum radicatorum ejusdem speciei.

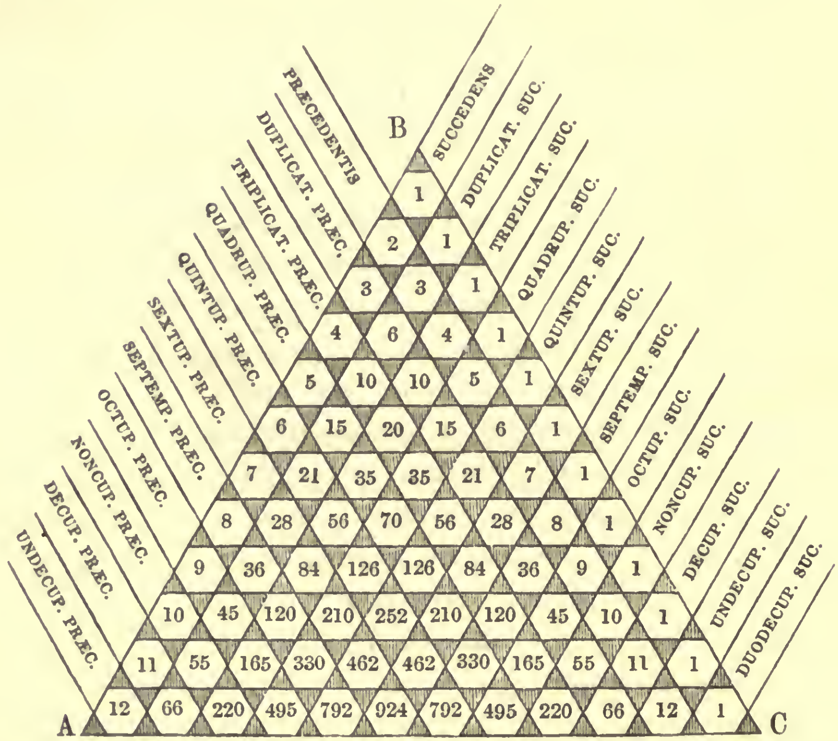
Ut sint radicata ejusdem speciei 100 et 144; videlicet, illud duplicatum 10, et hoc duplicatum 12; differentia duplicatorum 100 et 144 est 44, quæ sunt verum supplementum præfatorum radicatorum.

Supplementa ergo variantur, prout radicatorum et radicum species variæ fuerint. Alia enim est regula inveniendi supplementa duplicationis, et extractionis radicis bipartientis, alia triplicationis, et extractionis radicis tripartientis, alia quadruplicationis, et extractionis quadrupartientis radicis; et ita reliquarum omnium.

Regulas, autem, inveniendi omnium radicatorum et radicum supplementa, docet Tabella nostra triangularis, areolis hexagonis referta, dextrorsum inscriptis unitate solâ, sinistrorsum verò numeris ab unitate unitatis incremento crescentibus, et à vertice descendentibus; et quæ introrsum habet singularum areolarum numerum, quemque æqualem duobus numeris proxime superpositis.

Sit triangulum ABC, angulos, A sinistrum, B verticalem, et C dextrum, habens. Quot autem radicum species desideras tabellam comprehendere, per bis tot partes, plus unâ, dividatur quodque latus; v. g., ad duodecim extractionum species continendum, dividatur quodque latus in 25 æquales partes; et, incipiendo à basi AC, per singula alterna puncta laterum ducantur, intra triangulum, lineæ duodecim basi parallelæ; simili modo incipies à latere AB, et ei duodecim parallelas, à singulis alternis punctis basis, per singula alterna puncta lateris BC, tam intra triangulum quàm ultra lineam BC, digiti spatio extendes; eodem prorsus modo, lateri BC, duodecim parallelas à singulis alternis punctis basis, et per singula alterna puncta lateris BA, ultra triangulum digiti spatio extendes. Hinc habes triangulum areolis hexagonis refertum, quarum 12 dextimæ, et lineæ BC proximæ, 12 unitatibus sigillatim inscribuntur; sinistimæ verò numeris 1, 2, 3, 4, 5, etc., usque ad 13, ordine, à vertice B ad angulum sinistrum A descendentibus, inscribuntur; deinde hexagonum quodque interius vacuum inscribatur aggregato duorum numerorum ei proxime superpositorum; ut sub 2 et 1 scribantur 3; sub 3 et 3, 6; sub 3 et 1, 4; et ita in calcem usque tabulæ. Tandem, ascribantur tituli sinistrorsum, 'præcedentis,' supra secundum hexagonum 2; supra tertium 3 scribe, 'duplicatum præcedentis;' supra quartum scribe, 'triplicatum præcedentis;' et sic de cæteris radicatis præcedentis usque ad duodecuplicatum: dextrorsum verò scribe supra primum hexagonum 'succedens;' supra secundum, 'duplicatum succedentis;' supra tertium, 'triplicatum succedentis;' et sic de reliquis radicatis succedentis, usque ad tredecuplicatum, prout in tabellæ ipsius diagrammate subscripto habes.

TABULA SUPPLEMENTORUM.



Cuique supplemento respondent duæ partes radicis : alterâ constante unâ vel pluribus figuris sinistimis jam inventis, quæ præcedens dicitur ; alterâ constante unicâ dextrâ figurâ jam proxime inveniendâ, quæ succedens nuncupatur.

Atque hæ radicis partes, et supplementum, mutuo se componunt et condunt ad invicem ; ut postea patebit.

Ut duplicati 144 radix bipartiens est 12, quorum duæ partes,

1 et 2, respondent supplemento 44, et anterior 1 dicitur 'præcedens,' posterior verò 2, 'succedens;' at si duplicatum 15129 sumpseris, cujus radix bipartiens est 123, anterior pars jam nuper inventa, scilicet 12, dicitur 'præcedens,' et unica figura 3, 'succedens.'

Ex hâc itaque tabellâ, cujusvis supplementi inveniendi regula sic colligitur et legitur. Primò, quæratur juxta crus sinistrum numerus indicis propositæ radicis; hic enim, una cum reliquis arealibus numeris directe in eâdem lineâ ordine sequentibus, quæsitum supplementum legendum offert; modo una cum eorum singulo legantur tituli ei oblique, tam sinistrorsum quàm dextrorsum, superpositi invicem multiplicandi.

Exempli gratiâ:—Quæritur, è tabulâ, regula inveniendi supplementum duplicationis et extractionis radicis bipartientis. Hæc in secundâ lineâ duobus numeris, 2 et 1, et suis titulis invicem multiplicandis, exprimitur hoc modo,—præcedentis 2 succedens, et 1 duplicatum succedentis, quæ ita legenda et pronuntianda sunt: supplementum duplicationis constat præcedentis duplo multiplicato per succedens, et unico seu ipso duplicato succedentis; ut exemplis sequentibus patebit. Sic pro supplemento triplicationis, in tertiâ lineâ reperies tres numeros, 3, 3, 1, qui, cum suis titulis, ita leguntur: supplementum triplicationis tribus constat numeris; primus est, duplicati præcedentis triplum multiplicatum per succedens; secundus est, præcedentis triplum multiplicatum per duplicatum succedentis; tertius est, ipsum triplicatum succedentis.

Similiter, pro supplemento quintuplicationis, vel extractionis radicis quinquepartientis: In quintâ lineâ reperies quinque numeros hos, 5, 10, 10, 5, 1, qui, cum suis titulis, indicant

supplementum quintuplicationis, aut quintupartientis radicis, quinque constare partibus; quarum prima est quadruplicati præcedentis quintuplum ductum per succedens; secunda est, triplicati præcedentis decuplum ductum per duplicatum succedentis; tertia est, duplicati præcedentis decuplum ductum per triplicatum succedentis; quarta est, præcedentis quintuplum ductum in quadruplicatum succedentis; quinta est, ipsum quintuplicatum succedentis: Atque ita quadruplicationis, sextuplicationis, et reliquarum supplementa ex hâc tabellâ invenire poteris.

Ex sinistimarum figurarum radice quàm maximâ præcedente, et singulorum supplementorum figuris succedentibus in unum collectis, emerget radix plene extracta.

Supplementa autem omnium radicum hactenus descripta habes; superest ergo jam de earundem extractione disserere.

CAPUT VIII.

DE RADICUM EXTRACTIONE.

Omnis radicati radix aut est unius figuræ, aut plurium.

Radices omnes unicæ figuræ, infra tredecupartientem, exhibet tabella subsequens.

Ut si radicem quintupartientem 16809 quæras: In lineâ quintuplicatorum quærendus est oblatus numerus, aut ei saltem proxime minor, 16807; et loco eis directe supremo reperies figuram quæsitam, nimirum, radicem quintupartientem oblati, vel oblato proximi, quæ est 7, et supersunt 2, extractione irresolubiles.

TABULA RADICATORUM ET RADICUM.

	2	3	4	5	6	7	8	9
Duplicatum	4	9	16	25	36	49	64	81
Triplicatum	8	27	64	125	216	343	512	729
Quadruplicatum	16	81	256	625	1296	2401	4096	6561
Quintuplicatum	32	243	1024	3125	7776	16807	32768	59049
Sextuplicatum	64	729	4096	15625	46656	117649	262144	531441
Septemplicatum	128	2187	16384	78125	279936	823543	2097152	4782969
Octuplicatum	256	6561	65536	390625	1679616	5764801	16777216	43046721
Noncuplicatum	512	19683	262144	1953125	10077696	40353607	134217728	387420489
Decuplicatum	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
Undecuplicatum	2048	177147	4194304	48828125	362797056	1977326743	8589934592	31381059609
Duodecuplicatum	4096	531441	16777216	244140625	2176782336	13841287201	68719476736	282429536481

Construitur hæc tabula, ex continuatis multiplicationibus, per singulas novem figuras, seriebus ab eisdem figuris descendentibus.

Ut in fronte tabulæ, habes figuras, octo multitudinis, 2, 3, 4, 5, &c. usque ad 10; et sub quâque figurâ, suum duplicatum, triplicatum, quadruplicatum, &c. in tredecuplicatum usque: ut, sub 2 habes 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, cum suis inscriptionibus à margine sinistrorsum; nimirum, duplicatum, triplicatum, quadruplicatum, &c. et duodecuplicatum ultimò; sub 3, autem, habes sua radicata 9, 27, 81, 243, &c. continuatâ ratione descendentia; sub 4, item, habes 16, 64, 256, &c.: et ita ad novenarium usque; sub quo habes 81, 729, 6561, &c.; ut in tabulâ ipsâ perspici possunt.

In extractione radicis constantis pluribus figuris, spectandi sunt situs, et operatio.

Situs est, ut sub radicato oblato duæ ducantur lineæ parallelæ, quarum intervallum sit radicis quæsitæ capax, inter quas, sub figurâ radicis dextimâ, signetur punctum; abhinc autem sinistrorsum, sub secundâ quâque figurâ pro radice bipartiente, et sub tertiâ quâque pro tripartiente, et sub quintâ quâque pro quintupartiente, &c., notentur puncta.

In hæc puncta, inter lineas posita, cadunt figuræ singulæ radicis quæsitæ.

Operationis autem duo sunt præcepta.

Primum, ut à figuris, in sinistrum punctum terminantibus, auferas radicatum propositæ speciei, in tabulâ secundâ inventæ, quàm maximum auferre poteris, notatis supra reliquiis, et loco hujus sinistimi puncti ponatur assumpti radicati figura tabularis, pro primâ radicis figurâ.

Secundum, ut, factâ conjecturâ ex primis supplementi numeris, succedat in sequenti puncto, seu periodo ejusmodi, recens et nova figura, cujus supplementum propositæ speciei, quàm maximum sit, non tamen excedat figuras illi superiores in ipsâ hâc periodo terminantes; ex quibus inde supplementum idem auferatur, reliquiis supra notatis. Atque hæc secunda operatio toties, seu tot vicibus, repetatur, quot supersunt periodi vacuæ versus dextram; et figuræ in periodos incidentes sunt radix quæsita.

Ut, exempli gratiâ, sit extrahenda radix bipartiens è *55225*: Constituantur situ quo à margine; deinde primò, à *5*, sinistimæ periodi, aufer maximum duplicatum tabulæ quod auferri possit, scilicet 4, et restat 1, deletis *5* supra et 4 infra; quaternarii radix bipartiens tabularis, quæ est *2*, ponatur primæ periodi loco; *2* ergo dicuntur præcedens figura; cui succedens est quærenda, et in secundâ periodo constituenda, cujus supplementum non excedat suprascriptas

```
55225
=.=.=.=

1
$\cancel{5}5225$
-.-.-.-
2
-------
$\cancel{4}$
```

figuras 152, factâ conjecturâ ex primâ supplementi parte, scilicet ex 2 jam inventis (quæ 20 sunt in hoc loco) duplatis et multiplicatis per inveniendam, id est, 40 multiplicatis per succedentem. Si autem quaternarius foret succedens, prior supplementi pars foret 160, quæ excedunt 152; rejecto ergo quaternario, eligimus ternarium in hâc periodo collocandum; cujus supplementum duplicationis, ex priore tabulâ, est 129 (scilicet ter 40, et ter 3); quo supplemento ex 152 oblato, supersunt hoc loco reliquiæ 23, seu ad dextram usque 2325, et pars præcedens radicis jam est 23, vel 230; iterum ergo repetatur secunda operatio, ad inveniendam novam succedentem figuram, dextimæ periodi loco ponendam; hujus novæ succedentis supplementum, quodcunque sit futurum, constat (per primam tabulam) præcedentium 230 duplo, scilicet 460, multiplicato in succedentem, et succedentis duplicato; at si 6 multiplicaveris per 460, excedunt 2325; rejectis ergo 6, computetur supplementum quinarii; eritque, ex tabulis, 2325, (scilicet quinquies 460, et quinquies 5) quibus ablatis ex totidem suprapositis, nihil superest; et radix bipartiens 235, jam plena et perfecta, emersit oblati numeri 55225.

20
2
40
3
120. Prior pars prioris supplementi.
129. Totum primum supplementum.

3
3
9. Posterior pars prioris supplementi.

123
55225
2 3
4
129

230
2
460
5
2300. Prior pars posterioris supplementi.
2325. Totum posterius supplementum.

5
5
25. Posterior pars posterioris supplementi.

0
123
55225
2 3 5

Aliud exemplum extractionis bipartientis radicis.

```
164860
 4 0
16
```

Sit extrahenda radix bipartiens è 164860, collocatis ut à margine: Figurarum primæ periodi, scilicet 16, radix bipartiens est 4, in illâ periodo locanda; horum 4 duplicatum, 16, est ex superioribus 16 auferendum, et supersunt 48 in secundam periodum usque, et 40 sunt figuræ præcedentes; jam si vel unitatem pro succedente elegeris, erit ejus supplementum 81, quæ non possunt ex 48 auferri; ergo figura succedens et nova, in secundâ periodo ponenda, erit 0, cujus supplementum est 0; quo ex 48 ablato, supersunt in ultimam periodum 4860: et jam 40 sunt figuræ præcedentes inventæ; et quæritur nova succedens, ultimæ periodi loco statuenda; ea erit necessariò 6, cujus supplementum (scilicet sexies duplum 400, et sexies 6) est 4836; quibus ex 4860 sublatis, restant 24, reliquiæ irresolubiles; et radix bipartiens quæsita, imperfecta tamen, est 406, oblati radicati 164860.

```
400                   6
2                     6
800                   36. Posterior
6                     pars hujus
4800. Prior           supplementi.
pars hujus            4836. Totum
supplementi.          supplementum.
```

```
 0  24
164860
 4 0 6
16
  4836
```

Exemplum extractionis tripartientis radicis.

Sit extrahenda radix tripartiens è 12977875: punctis et situ collocentur ut à margine; inde, maximum triplicatum tabulare non excedens 12, scilicet 8, ex 12 aufer, et supersunt 4, superscribenda, deletis 12; octonarii autem radix tripartiens, scilicet 2, primæ periodi loco statuatur, et 20 sunt jam præcedens; cujus succedens,

```
 4
12977875
 2
 8
```

cum suo supplemento, sic quæratur : supplementum tripartientis radicis (ex primâ tabulâ) constat tribus partibus, quarum prima fit ex triplo duplicati præcedentis multiplicato per succedentem ; unde sequitur, si 4977, suprascripta, dividerentur per triplum duplicati præcedentis, scilicet per triplum 400, quod est 1200, de succedente dabit quotus conjecturam ; succedens enim plerumque est aut huic quoto æqualis, aut unitate eo minor ; si itaque 4977 per 1200 partirentur, 4 foret quotus et succedens ; at quaternarii supplementum est 5824, quæ excedunt 4977 ; rejecto ergo 4, eligamus 3, quorum supplementum est 4167 (nam 1200 ducta per 3, et ter 3 ducta per ter 20, et triplicatum 3, aggregata faciunt 4167) ; his ex 4977 ablatis, supersunt hic 810, et in periodum tertiam 810875, et 230 jam sunt figuræ præcedentis. Quærenda restat succedens in dextimum punctum locanda : ea, simili modo quo præmissa, invenietur ; estque 5, cujus supplementum est 810875 (nam triplum duplicati 230 ductum per 5, scilicet 793500, et

```
  20
  20
----
 400
   3
----
1200
   3
----
3600. Prima pars prioris supplementi.

 20    3
  3    3
---   --
 60    9
  9
---
540. Secunda pars prioris supplementi.

  3
  3
---
  9
  3
---
 27. Tertia pars prioris supplementi.

3600
 540
  27
----
4167. Totum prius supplementum.
```

~~4~~810
~~12977~~875
2 3
~~8~~
~~4167~~

```
   230
   230
------
    69
   46
------
 52900
     3
------
158700
     5
------
793500. Prima pars posterioris supplementi.

  230
    3
-----
  690
   25
-----
  345
 138
-----
17250. Secunda pars posterioris supplementi.
```

5 5 25 5 125. Tertia pars posterioris supplementi.	793500 17250 125 810875. Totum posterius supplementum.

~~1~~8~~1~~~~0~~
~~1~~~~2~~~~9~~~~7~~~~7~~~~8~~~~7~~~~5~~
2 3 5
~~8~~
~~4~~~~1~~~~6~~~~7~~
~~8~~~~1~~~~0~~~~8~~~~7~~~~5~~

triplum 230 ductum in 25, videlicet 17250, et triplicatum 5, quod est 125, fiunt 810875); his ergo ex superpositis eis æqualibus ablatis, nihil superest; et radix tripartiens quæsita, 235, perfecta et integra, oblati triplicati 12977875, producitur. Atque ita in omnibus extractionibus radicum, ubi indices sunt primi et incompositi, ut quintupartientis, et sextupartientis, &c., operaberis.

Exemplum extractionis radicum ubi indices sunt compositi.

~~5~~~~5~~~~2~~~~2~~
~~2~~~~7~~~~6~~ 0
~~5~~~~2~~~~4~~
~~3~~~~0~~~~4~~~~9~~~~8~~~~0~~~~0~~~~6~~~~2~~~~5~~
5 5 22
~~2~~~~5~~ ~~5~~~~5~~~~2~~~~2~~~~2~~~~5~~
~~5~~~~2~~~~5~~
~~2~~~~2~~~~0~~~~4~~
~~2~~~~2~~~~0~~~~8~~~~4~~

Sit extrahenda radix quadrupartiens è 3049800625; quòd hoc expeditius fiat, bis extrahendo radicem bipartientem, quàm semel extrahendo radicem quadrupartientem, consule cap. 4 Lib. I. Eorum ergo extrahatur radix bipartiens, situ et operatione debitis, ut à margine, et producentur 55225, radix bipartiens oblatorum; rursus hujus radicis bipartientis extrahe jam secundò radicem bipartientem, ut

50 2 100 5 500	5 5 25. Supplementum primum.
550 2 1100 2 2200	2 2 4. Supplementum secundum.
5520 2 11040 2 22080	2 2 4. Supplementum tertium.
55220 2 110440 5 552200	5 5 25. Supplementum quartum.

0
23
1
55225
$\dot{2}\ \dot{3}\ \dot{5}$
4
120
2325

etiam à margine vides, et reperies hujus bipartientis bipartientem radicem esse 235, quæ etiam est radix quadrupartiens optata, radicati 3049800625 oblati. Atque ita in extractione radicum sextupartientium, octupartientium, nonupartientium, et aliarum quarumcunque compositarum, operaberis.

20		
2		
40	3	
3	3	
120	9.	Supplementum primum.
230		
2		
460	5	
5	5	
2300	25.	Supplementum secundum.

CAPUT IX.

DE RATIONE EMENDANDI EXTRACTIONES IMPERFECTAS.

Quæ post extractionem supersunt reliquiæ, quantumcunque irresolubiles quoad totum, tamen ex parte seu partim reformari possunt.

Hæ autem superstites reliquiæ, nisi ex parte seu partim reformentur, errorem plerumque sensibilem pariunt.

Sensibilis radicis et suarum reliquiarum error duobis modis præcipuis emendatur.

Prior est, post radicem imperfectam ducendo lineam, supra quam scribantur reliquiæ, infra verò supplementum unitatis, integer pro minore termino, et unitate minutus pro majore; inter hos enim terminos latet vera quantitas radicis, quæ numero nulli definiri possit.

Ut sit extrahenda radix bipartiens è 164860, in secundo præcedentium exemplo reperies eorundem radicem bipartientem

406 imperfectam, et reliquias 24 superstites. Constitue ergo lineam post 406, et supra lineam scribantur 24, ut à margine; inde ex inventâ jam præcedente 406, quære supplementum unitatis succedentis respondens duplicationi; hoc integrum erit 813, et unitate minutum erit 812, quorum illud pro minore, hoc pro majore termino bis sub lineâ scribatur. Erit ergo quæsitæ radicis bipartientis minor terminus $406\frac{24}{813}$, et major $406\frac{24}{812}$, inter quos terminos continetur et latet præcisa radix bipartiens numeri 164860 oblati. Ita ut absque sensibili errore (præcipue in mechanicis) radix bipartiens 164860 possit dici $406\frac{24}{813}$, vel $406\frac{24}{812}$.

$406\frac{24}{813}$

$406\frac{24}{812}$

406	
2	
812	1
1	1
812	1. Unitatis supplementum.

Radicis tripartientis emendandi exemplum.

Sit extrahenda radix tripartiens numeri 998, hæc (ex præcedente tabulâ) deprehenditur proxime esse 9, et supersunt adhuc 269 reliquiæ emendandæ. Illa 9 ante lineam, hæc 269 supra eam, ut à margine statuantur. Ex figurâ autem 9, pro præcedente inventâ, unitatis supplementum triplicationis integer reperietur 271, et unitate minutus 270, ut à margine conspicere licet, quæ sub lineis scribantur. Et ita radix tripartiens numeri 998 oblati præcise inter terminos $9\frac{269}{271}$ mino-

$9\frac{269}{271}$

$9\frac{269}{270}$

9	9	1
9	3	1
81	27	1. Supplementi tertia pars.
3	1	
243	27	27. Secunda pars.
1		
243		243. Prima pars.
		271. Totum supplementum unitatis.

rem, et $9\frac{269}{270}$ majorem, constituitur; ita ut pro eâ, hæc vel illa absque sensibili errore capi possit.

Posterior modus est, ut totum radicatum propositæ speciei oblatum per cujusvis electi numeri radicatum ejusdem speciei multiplicetur; producti autem radix ejusdem speciei (spretis reliquiis) per numerum illum electum partiatur. Nam quotus, cum suis fragmentis lineâ distinctis, erit minor terminus; et si ad eorundem fragmentorum numeratorem addideris unitatem, erit major terminus, inter quos continetur vera radix.

Ut sit extrahenda radix bipartiens emendata admodùm, è numero 50. Electus numerus sit 1000, ejus duplicatum erit 1000000; per quæ multiplica 50, fient 50000000 producta; quorum radicem ejusdem speciei, bipartientem nempe, quære, situ et operatione quibus à margine; et erit hæc radix bipartiens 7071 (neglectis reliquiis 959), per electum 1000 partire, et fiet inde quotus, cum fragmentis, $7\frac{71}{1000}$, pro minore termino, et $7\frac{72}{1000}$ pro majore termino. Ita ut alterutrum, pro ipsâ radice bipartiente numeri 50 oblati, capi possit, absque errore perceptibili.

```
     959
    151
  1
 50000000
 ————————
  7 0 7 1
 ————————
 49
   9849
    14141
```

```
  700
    2
 ————
 1400     7
    7     7
 ————    ——
 9800    49.  Supplementum primum.

 7070
    2
 —————
 14140    1
     1    1
 —————   ——
 14140    1.  Supplementum secundum.
```

Exemplum præcedentis tripartientis radicis hoc modo emendatæ.

Sit extrahenda tripartiens radix emendata, è numero 998. Sit alius numerus electus 100; ejus triplicatum erit 1000000;

per hæc multiplica 998, provenient 998000000, quarum extrahatur tripartiens radix, ut à margine, et ea producetur quàm proxime 999, quæ (spretis jam reliquiis 997001) per electum numerum 100 partiantur; proveniet inde $9\frac{99}{100}$, pro minore termino, et (numeratore unitate aucto) $9\frac{100}{100}$, quæ sunt 10, pro termino majore. Itaque, absque perceptibili errore, $9\frac{99}{100}$, vel 10, dici possunt radix tripartiens numeri 998 oblati; inter hos enim terminos arctissime includitur.

997001
27701
269
998000000
9 9 9
729
241299
26703999

	90	
90	3	
90	270	81
8100	81	9
3	270	729
24300	216	
9	21870	

218700.	Supplem. primi prima pars.
21870.	Secunda pars.
729.	Tertia pars.
241299.	Totum suppl. prius.

990		
990	990	
89100	3	
891	2970	81
980100	81	9
3	2970	729
2940300	2376	
9	240570	

26462700.	Posterioris supp. prima pars.
240570.	Secunda pars.
729.	Tertia pars.
26703999.	Totum posterius suppl.

Hi modi, quia radices imperfectas non perficiunt, sed nimis imperfectas reddunt, Mechanicis magis quàm Mathematicis placent. Ut cap. 4 Lib. I. diximus.

Geometræ ergo his radicatis numeris, radices non habentibus, præponunt signum radicis debitæ. Unde ex his ita signatis oritur prima species geometricorum numerorum, quam uninomia vocant.

Ut superiorum duplicatorum 164860, et 50, non extrahunt radices bipartientes (quia nullas præcise habent in numeris),

nec extractas emendant, sed signum radicis extrahendæ, quam quadratam vocant, numero præponunt, hoc modo, √♀164860, et, √♀50; vel sic, √q164860, et, √q50; quæ sic pronuntiant, radix quadrata numeri 164860,—et, radix quadrata numeri 50. Nos autem ita notamus, ⊔164860, et ⊔50; et sic pronuntiamus, radix bipartiens 164860,—et, radix bipartiens 50. Ita, radicem tripartientem numeri 998 nec extrahunt (quòd non sit in numeris) nec emendant, sed ita signant, √C998, et ita pronuntiant, radix cubica 998. Nos autem ita notamus, ∟998; proferimus autem sic, radix tripartiens 998, ut amplius suo loco dicemus. Utcunque, hæc uninomia seu medialia dicuntur, et sunt Geometricæ Logisticæ fundamentum: sequenti ergo Libro tractabuntur; hinc autem ea oriri monuisse satis est.

Hucusque computationes simplices numerorum integrorum; sequuntur compositæ, seu regulæ.

CAPUT X.

DE REGULIS PROPORTIONIS INTEGRORUM.

Proportionis integrorum numerorum regulæ, sub generali quantitatum methodo, et præceptis initio fuse satis explicantur.

Prout in tribus quæstionibus exemplaribus, simplicis, duplicis, et triplicis regulæ proportionis, cap. 5 Lib. I. propositis, et unicâ generali methodo solutis, cernere poteris.

Particularia tamen numerorum exempla specialia habent compendia, quibus expediantur: Si enim in majusculis numeris, partitor aliquot

habeat versus dextram circulos, poteris, compendii gratiâ, tot fere multipli loca dextrorsum à figuris vacua, aut circulis referta relinquere, multiplicationem à sinistris ordiendo.

Ut (exemplo è sinibus) si 10000000 dederint 9925461, quantum dabunt 7986354? At quia hæc omnes fere figuras complectitur, ideo, per cap. 5 Lib. II. compendiose multiplicentur 9925461 per singulas novem figuras, ut in tabulâ à margine factum est; deinde sub singulâ figurâ multiplicantis 7986354 incipiat, et inde procedat numerus tabulæ figuræ illi respondens, neglectis tamen et omissis sex dextimorum locorum figuris omnibus, propterea quòd septem circuli partitoris 10000000 eas abscinderent, si exprimerentur et non omissæ fuissent: Hæc igitur particularia multipla, præter sua sex loca dextima vacua, in unum addita fiunt 79268241 . . . . . . à quibus, tam locis quàm figuris dextimis, si abscindantur septem circulorum partitoris otiosa loca (ut partitionis compendium exigit), restabunt 7926824, responsum quæsitum. Ubi ergo 10000000 dant 9925461, sequetur, quòd 7986354 dabunt 7926824.

09925461	1
19850922	2
29776383	3
39701844	4
49627305	5
59552766	6
69478227	7
79403688	8
89329149	9
99254610	Decupl.

```
7986354
69478227
 8932914.
  794036..
   59552...
    2977....
     496.....
      39......
7926824|1......
        10000000
```

Quæ autem, in hujus compendii multiplis, omittuntur figuræ versus dextram, etiamsi omnes novenariæ essent, non vel unicâ unitate augerent responsum. Meritò igitur eæ omnes negligi possunt in his majusculis numeris in quibus ne unitatis quidem, totius et integri, error est sensibilis.

Esto enim quòd vacua illa puncta, à margine posita, novenariæ essent (quod supra possibilitatem est), nihilominus ad 5888889 tantum accrescerent, quæ ad 79268241. . . . . . addita facerent 79268246888889, quibus per 10000000 partitis proveniunt ad summum $7926824\frac{6888889}{10000000}$, quæ non extenduntur ad 7926825, nec superius productum unitate exuperant. In maximis itaque numeris, maximâ laude dignum est hoc compendium regulæ trium.

.
. .
. . .
. . . .
. . . . .
. . . . . .
5888889

Est etiam aliud compendium hujus regulæ, absque omissione figurarum, omnes oblatos quæstionis numeros debitis locis supra vel infra lineam constituendo, ut in generali methodo, cap. 5 Lib. I. proposita præcepimus; deinde singuli bini numeri, quorum alter est superior et quasi numerator, alter inferior quasi denominator, partiantur per suum maximum communem divisorem, donec singuli numeratores ad singulos denominatores fuerint in ratione primâ seu minimâ ad invicem, notatis omnium ultimis quotis; tandem, multiplum quotorum superiorum omnium partiatur per multiplum quotorum inferiorum, et hic quotus erit responsum quæsitum, quæstioni satisfaciens.

Ut 4 ædificantes construxerunt murum 6 pedes altum, 48 ulnas longum, diebus 42; quæritur, quot diebus 5 ædificantes ædificabunt murum 9 pedes altum, 50 ulnas longum? Per cap. 5 Lib. I. disponantur omnes numeri debito situ, et stabunt ut à margine; deinde numerum superiorem 4, et inferiorem 6, abbrevia per 2, eorum maximum divisorem, et fient $\frac{2}{3}$, hoc situ, $\frac{2\cdot 9\cdot 50\cdot 42}{5\cdot 3\cdot 48\cdot}$; deinde partire 2 superiorem, et 48 inferiorem, per communem partitorem 2, et fient 1, et 24, hoc situ, $\frac{1\cdot 9\cdot 50\cdot 42}{5\cdot 3\cdot 24\cdot}$; deinde partiantur superiora 9, et inferiora 3, per 3, et fient 3 supra, et 1 infra, hoc situ, $\frac{1\cdot 3\cdot 50\cdot 42}{5\cdot 1\cdot 24\cdot}$; deinde

Ædific.	Pedes.	Ulnæ.	Dies.
4	9	50	42
5	6	48	Quot dies?

partiantur 50, et 5, per 5, et fient superius 10, inferius 1, hoc situ, $\frac{1\cdot 3\cdot 10\cdot 42}{1\cdot 1\cdot 24\cdot}$; deinde partiantur 10 superiora, et 24 inferiora, per suum maximum communem divisorem 2, et fient 5 supra, et 12 infra, hoc situ, $\frac{1\cdot 3\cdot \ 5\ \cdot 42}{1\cdot 1\cdot 12\cdot}$; tandem, partiantur superiora 42, et inferiora 12, per eorundem maximum divisorem 6, et fient 7 superius, et 2 inferius, hoc situ, $\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 1\cdot 2\cdot}$. Ecce jam habes familiares et tractabiles numeros 1, 3, 5, 7, atque 1, 1, 2, invicem multiplicandos, pro majusculis numeris oblatis. Ducantur ergo invicem 1, 3, 5, 7, et fient 105; ducantur item inferiora 1, 1, 2, invicem, et fient 2; per quæ partiantur 105, et proveniet quotus $52\frac{1}{2}$, numerus dierum quæstioni satisfaciens, absque magnis aut laboriosis multiplicationibus et divisionis operibus.

Hæc proportionum regula plurimas habet tacitas et latentes species, non negligendas, quæ beneficio etiam hujus compendii fruuntur, et eo expediuntur.

Ut regula reducendi fractiones fractionum, iterum atque iterum fractarum (de quâ generaliter superius, et particulariter inferius tractabitur), est species hujus regulæ proportionis, ex definitione regulæ proportionis. Unde eodem compendio illa, quo hæc, abbreviatur, ut in exemplis cap. 8 Lib. I. ostensum est, et inferius ostendetur.

De integrorum numerorum Arithmeticâ satis; superest fractorum, seu fractionum.

CAPUT XI.

DE FRACTIONIBUS, SEU NUMERIS FRACTIS.

Fractio, seu numerus fractus, est quem minima et individua pars, unica scilicet unitatis numerata, metitur.

> Ut apud astronomos gradus pro unitate habetur, et 7 scrupula prima sunt ejus fractio; nam hoc loco unicum scrupulum primum est ejus minima et individua pars, quæ eam metitur, et ab eâ numeratur per 7. Sic 5 solidi sunt libræ fractio, quia 1 solidus hîc est minima et individua unius libræ pars, quæ fractionem metitur, et ab eâ numeratur per 5. Sic 17 trigesimæ unitatis partes sunt fractio; quia unica trigesima est hoc loco minima et individua unitatis pars, quæ 17 trigesimas metitur, et ab eis numeratur septemdecies.

Fractionum, aliæ vulgares, aliæ physicæ dicuntur.

Vulgares sunt, quæ varias et liberas habent denominationes.

> Ut unum dimidium, duæ tertiæ, quatuor undecimæ, quæ ortum habent in partitione, cap. 4 Lib. II.

In Arithmeticâ fractionum vulgarium spectantur nominatio et operatio.

Nominatio, quæ voce fit, dicitur pronuntiatio; quæ autem scriptis, notatio.

Vulgarium fractionum nominatio duobus terminis exprimitur, numeratore et denominatore.

Denominator est, qui nominat in quot partes æquales unitas sit distribuenda.

Numerator verò est, qui numerat quot ex his partibus sint sumendæ.

Atque hic prius, et cardinali numero,—ille posterius, et ordinali, pronuntiatur; hic quoque supra lineam, ille verò infra, notatur.

Ut si unitatem, aut rem unicam propositam, in partes quotvis, ut v. g. in 11, partitus fueris, et harum 5 partes sumendas esse statuas—11 sunt hujus fractionis denominator, et 5 sunt ejusdem numerator; quorum, hæc 5 prius, et per numerum cardinalem (quinque), profertur; et illa 11 posterius, et per numerum ordinalem (undecimas), pronuntiatur; hoc modo, 'quinque undecimas partes.' Notantur verò hæc 5 supra lineam, et illa 11 infra, hoc modo, $\frac{5}{11}$; et ita de aliis.

Huic constat quòd fractio, seu fractus numerus, sit pars, aut partes, unitatis divisæ in partes aliquot.

Constat etiam quòd fractio idem valet quod numerus numeratoris, divisus per numerum denominatoris.

Ut in superiore fractione $\frac{5}{11}$, idem est eam pronuntiare, 'quinque undecimas unitatis,' quod, 'quinque partita per 11;' ut in cap. 7 Lib. I. generaliter diximus.

Sunt et quædam improprie fractiones, quæ non sunt unitatis pars, aut partes, expresse, sed sunt partes fractionum; et hæ fractiones fractionum nominantur; quas nos notamus interpositâ particulâ 'ex,' alii notant per omissionem posterioris lineæ, aut linearum.

Ut duæ quintæ partes, trium quartarum partium unitatis, non est proprie, nec immediate, unitatis fractio, sed fractio fractionis unitatis; significant enim unitatem in quatuor partes dividi, et harum tres partes rursus in 5 partes partiri, quarum duas tandem sumendas esse. Ideo sic notamus, $\frac{2}{5}$ ex $\frac{3}{4}$, alii sic notant, $\frac{2}{5}\,\frac{3}{4}$; et sic pronuntiamus, 'duæ quintæ ex tribus quartis,' aut, 'duæ quintæ trium quartarum.'

Vulgarium fractionum nominationem sequuntur operationes, et computationum praxes.

Ergo, quoad praxin, reducimus integrum ad fractionis speciem, subjiciendo unitatem pro denominatore, et abbreviamus fractionum terminos quum accreverint, partiendo eos per suum maximum communem divisorem, quem etiam invenimus continuâ partitione partitoris per suas reliquias, donec nihil remanserit; atque hujus beneficio diversas denominationes ad eandem reducimus; reductas tandem addimus et substrahimus, in omnibus et per omnia, juxta canones quantitatum in genere cap. 7 Lib. I. conscriptos.

Ut binarius, numerus integer, reducitur ad speciem fractionis cum fit $\frac{2}{1}$. Sic 3 ad $\frac{3}{1}$, et 4 ad $\frac{4}{1}$, et 5 ad $\frac{5}{1}$, idem enim valent. Abbreviamus etiam terminos hujus fractionis, $\frac{6}{10}$, partiendo et numeratorem et denominatorem per suum maximum communem divisorem 2, et fient $\frac{3}{5}$. Sic $\frac{35}{49}$, partitæ per suum maximum communem divisorem 7, fiunt $\frac{5}{7}$. Quem etiam maximum divisorem ibidem sic invenies: partire 49 per 35, remanent 14; partire 35 per 14, remanent 7; partire 14 per 7, remanent nihil; 7 ergo sunt maximus communis divisor terminorum 35 et 49. Sic etiam per eosdem canones, duas fractiones diversarum denominationum, ut $\frac{11}{132}$ et $\frac{7}{128}$, ad eandem denominationem reduces hoc modo: partire 132 et 128 per suum maximum divisorem communem, scilicet per 4, et fient 33 et 32; duc 7 per 33, et fient 231; et duc 128 per 33, et fient 4224, hoc situ, $\frac{231}{4224}$, pro fractione $\frac{7}{128}$ oblatâ: Similiter, duc 32 per 11, et 32 per 132, et fient 352 et 4224, hoc situ, $\frac{352}{4224}$, pro fractione $\frac{11}{132}$ oblatâ. Habes ergo duas oblatas fractiones, $\frac{11}{132}$ et $\frac{7}{128}$, ad has, $\frac{352}{4224}$, et $\frac{231}{4224}$, ejusdem denominationis, reductas, per dictos generales canones. Quibus jam reductis, per eosdem canones addimus dictos numeratores, et fiunt, cum communi denomina-

tore, $\frac{583}{1221}$, pro additionis toto. Sic, substrahimus numeratorem novum 231 ex novo 352, et supersunt, cum novo denominatore, $\frac{121}{1221}$, pro substractionis residuo. Item, sint reducendæ ad eandem denominationem $\frac{4}{12}$ et $\frac{7}{15}$: primò abbreviantur, fiuntque $\frac{1}{3}$ et $\frac{7}{15}$; deinde partiantur 3 et 15 per maximum communem divisorem 3, et fient 1 et 5; duc $\frac{1}{3}$, tam numeratorem quàm denominatorem, per 5, et fient $\frac{5}{15}$; sic duc $\frac{7}{15}$ per 1, et manent $\frac{7}{15}$, ejusdem denominationis cum $\frac{5}{15}$; quibus jam additis, fient $\frac{12}{15}$, sive, per abbreviationem, $\frac{4}{5}$, pro additionis toto. Simili modo, pro substractionis residuo, aufer $\frac{5}{15}$ à $\frac{7}{15}$, et relinquentur $\frac{2}{15}$, substractionis residuum. Item, sint addendæ, seu potius uniendæ, 66 et $\frac{2}{3}$: primò fient $\frac{66}{1}$ et $\frac{2}{3}$, per unitatis subscriptionem; deinde, per reductionem ad communem denominatorem, fient $\frac{198}{3}$ et $\frac{2}{3}$; tandem, additione numeratorum, fient $\frac{200}{3}$. Item, sint addendæ fractiones $\frac{1}{2}$ et $\frac{2}{3}$ et $\frac{3}{4}$: hæ, per reductionem ad eandem denominationem, fient primò $\frac{6}{12}$ et $\frac{8}{12}$ et $\frac{9}{12}$; deinde, suprascribendo aggregatum numeratorem, et subscribendo communem denominatorem, fient $\frac{23}{12}$, quæ sunt 1 et $\frac{11}{12}$. Item, sint auferendæ $\frac{8}{12}$ ex $\frac{11}{12}$: restant primò, per substractionem, $\frac{3}{12}$; deinde, per abbreviationem, fient $\frac{1}{4}$, residuum quæsitum.

CAPUT XII.

DE MULTIPLICATIONIBUS ET PARTITIONIBUS SIMPLICIBUS ET RADICALIBUS FRACTORUM NUMERORUM.

Multiplicantur fracti, atque etiam fractiones fractionum iterum atque iterum fractarum ad simplices fractiones reducuntur, singulum quemque numeratorem in singulum quemque denominatorem, per

cap. 7 Lib. I., ita abbreviando ut nulla remaneat inter eos composita ratio. Inde, ductis novis superioribus invicem, fit novus numerator; et novis inferioribus invicem, fit novus denominator, multipli aut fractionis quæsitæ.

Ut sint $\frac{1078}{1768}$ et $\frac{3705}{1449}$ et $\frac{1455}{2090}$ invicem multiplicanda: abbrevientur superiora 1078, et inferiora 1768, per 2, et fient 539 supra, et 884 infra. Item, abbrevientur 3705 superiora, et 1449 inferiora, per 3, et fient 1235 superius, et 483 inferius. Inde, abbrevientur 1455 et 2090, per 5, et fient 291 superius, et 418 inferius; hoc situ, $\frac{539}{884}$ $\frac{1235}{483}$ $\frac{291}{418}$: deinde abbrevientur 539 superius, et 483 inferius, per suum communem divisorem 7, et fient 77 supra, et 69 infra; deinde abbrevia 1235 superius, et 418 inferius, per suum communem divisorem 19, et fient 65 supra, et 22 infra; deinde abbrevia 291 superius, et 884 inferius, per suum maximum communem divisorem 17, et fient 23 superius, et 52 inferius; hoc situ, $\frac{77}{52}$ $\frac{65}{69}$ $\frac{23}{22}$: His factis, abbrevia 77 superius, et 22 inferius, per suum communem maximum divisorem 11, et fient 7 superius, et 2 inferius; sic abbrevia 65 superius, et 52 inferius, per suum communem divisorem maximum 13, et fient 5 supra, et 4 infra; tandem, abbrevia 23 superius, et 69 inferius, per suum communem partitorem maximum, videlicet per 23, et fient 1 superius, et 3 inferius, hoc situ, $\frac{7 \cdot 5 \cdot 1 \cdot}{4 \cdot 3 \cdot 2 \cdot}$: hos itaque faciles et tractabiles numeros superiores, 7, 5, 1, invicem multiplica, et fient 35 pro novo numeratore; similiter, multiplicentur invicem inferiora 4, 3, 2, et fient 24 pro novo denominatore: Est ergo fractio $\frac{35}{24}$ multipla quæsita omnium oblatarum fractionum. Item, sint invicem multiplicandæ hæ inabbreviabiles fractiones $\frac{3}{4}$ et $\frac{5}{7}$: multiplicentur ergo 3 in 5, et fient 15, novus numerator; multiplicentur etiam inferiores 4 in 7, et fient 28, novus denominator: Unde $\frac{15}{28}$ sunt fractio

quæsita multupla. Item, sint reducendæ $\frac{2}{3}$ ex $\frac{4}{5}$ ex $\frac{6}{7}$ ex $\frac{8}{9}$ ad eandem simplicem fractionem: abbrevientur primò 6 superius, et 9 inferius, per suum communem maximum divisorem 3, et fient 2 supra, et 3 infra, hoc situ, $\frac{2}{3}\ \frac{4}{5}\ \frac{2}{7}\ \frac{8}{3}$; ulterius enim non est inter has superior qui sit cum inferiore aliquâ abbreviabilis: multiplicentur ergo, tandem, superiores 2, 4, 2, 8, invicem, et fient 128; itemque, multiplicentur 3, 5, 7, 3, inferiora invicem, et fient 315; ex his fiet novus denominator, ex illis verò numerator, hoc situ, $\frac{128}{315}$, fractio simplex quæsita, idem valens quod $\frac{2}{3}$ ex $\frac{4}{5}$ ex $\frac{6}{7}$ ex $\frac{8}{9}$ oblatæ.

Pro partitione autem transponantur termini partitoris, et transpositos per partiendum multiplica omninò, ut in præcedente, et cap. 7 Lib. I., præcepimus.

Ut si partiendæ sint $\frac{4}{5}$ per $\frac{2}{3}$: retentis illis, inverte has, et ita multiplicato $\frac{4}{5}$ et $\frac{3}{2}$; fientque, primò, per abbreviationem, $\frac{2}{5}$ et $\frac{3}{1}$; deinde fient, per multiplicationem, $\frac{6}{5}$, sive, quod idem est, $1\frac{1}{5}$, pro quoto quæsito.

Aliud exemplum mistarum.

Sint $66\frac{2}{3}$ partiendæ per $2\frac{1}{5}$, seu, quod idem est, per reductionem partiantur $\frac{200}{3}$ per $\frac{11}{5}$: has inversas hoc modo, $\frac{5}{11}$, et illas, $\frac{200}{3}$, invicem multiplica, et provenient $\frac{1000}{33}$, quæ sunt $30\frac{10}{33}$, pro quoto quæsito.

Aliud exemplum, pro examine multiplicationis.

Sint, ex præcedentibus exemplis, $\frac{15}{28}$ partiendæ per $\frac{3}{4}$: invertatur hæc, fiet $\frac{4}{3}$, et illa manet $\frac{15}{28}$; abbrevientur, et fient primò

$\frac{5}{28}$ et $\frac{4}{1}$, deinde fient $\frac{5}{7}$ et $\frac{1}{1}$; hos numeratores invicem, et denominatores invicem duc, et fient $\frac{5}{7}$, ut superius in multiplicatione.

Multiplicatio radicalis in fractis numeris, prout in integris, per continuatam multiplicationem perficitur: Si enim fractionem oblatam in se, aut sibi æqualem, multiplicaveris, fit duplicata; si hanc duplicatam in eandem primam multiplicaveris, fit triplicata; quam triplicatam si adhuc in primam multiplicaveris, proveniet inde quadruplicata radicis et fractionis primò oblatæ: Et sic deinceps quintuplicata, sextuplicata, et ceteræ radicatæ componuntur.

Ut sit radix hæc, $\frac{2}{3}$, radicaliter multiplicanda ad indicem 5, seu quintuplicanda: Primò, multiplica $\frac{2}{3}$ in $\frac{2}{3}$, fient $\frac{4}{9}$ pro duplicatâ; deinde multiplica hanc duplicatam $\frac{4}{9}$ per easdem $\frac{2}{3}$, fient $\frac{8}{27}$, triplicata oblatæ fractionis; hanc triplicatam $\frac{8}{27}$ multiplica per easdem $\frac{2}{3}$, et proveniet oblatæ radicis quadruplicata $\frac{16}{81}$; quam tandem per $\frac{2}{3}$ multiplica, et producentur $\frac{32}{243}$.

Partitio etiam radicalis fractionum, ex continuatâ partitione simplici, per regulas superiores, et integrorum more, perficitur, partiendo fractionem radicatam propositam per suam radicem datam, continue, in unitatem usque; et numerus partitionum dabit indicem, ut superius sæpe dictum est.

Sint, pro exemplo et examine præcedentis, radicaliter partiendæ $\frac{32}{243}$ per $\frac{2}{3}$: ea fient, primò, $\frac{16}{81}$; secundò, $\frac{8}{27}$; tertiò, $\frac{4}{9}$; quartò, fiet ipsa radix $\frac{2}{3}$; quintò, devenit ad unitatem: Itaque, quinque hæ partitiones exhibent indicem quinarium pro radicali quoto quæsito, et arguunt præcedentem quintuplicationem legitimam esse.

CAPUT XIII.

DE EXTRACTIONE RADICUM E NUMERORUM FRACTIONIBUS.

Si, fractionis reductæ ad minimos terminos, uterque terminus habuerit radices propositæ speciei, eæ extrahendæ sunt integrorum more, et fient numerator et denominator radicis quæsitæ.

Sin autem uterque terminus fractionis non habeat talem radicem, tunc ejus radix, aut geometricæ more aut mechanico, est extrahenda.

Ut sit radix tripartiens extrahenda è $\frac{54}{16}$, cujus termini minimi sunt $\frac{27}{8}$; in hujus termino utroque clare constat radix tripartiens numeratoris 3, et denominatoris 2; sunt ergo $\frac{3}{2}$ radix tripartiens quæsita oblati radicati $\frac{54}{16}$, seu $\frac{27}{8}$.

Aliud exemplum.

Contra verò sit fractio oblata $\frac{290}{40}$, cujus termini minimi sunt $\frac{29}{4}$; sit ex his extrahenda radix bipartiens: Dico, quia uterque terminus, 29 et 4, non habeat radicem bipartientem, (quamvis alter, scilicet 4, habeat) ideo, fractionis oblatæ $\frac{290}{40}$, seu $\frac{29}{4}$, non est arithmetice extrahenda radix perfecta, sed aut geometrice aut mechanice.

Geometricus modus est, præponere radicatæ fractioni toti signum radicis extrahendæ, aut signa radicis utrique termino.

Ut superioris exempli fractio $\frac{290}{40}$, seu $\frac{29}{4}$, veram radicem bipartientem non habet; ideo, toti fractioni aut præponendum est signum radicis, hoc modo, $\sqcup\frac{29}{4}$, aut utrique termino, hoc

modo, $\frac{\sqcup 29}{\sqcup 4}$; aut, tandem, eadem radix est mechanice admodum inter terminos includenda, ut sequitur.

Mechanicus autem modus est, radicem quæsitam, quæ numeris exprimi nequit, inter terminos majorem et minorem quàm arctissime includere.

Termini quanto majoris denominationis, et minoris differentiæ fuerint, tanto radicem arctius includunt, et præcisius definiunt.

Ut ergo termini sint denominationis optatæ, numerum optatæ denominationis multiplica radicaliter, secundum indicem et speciem radicis quæsitam; inde, hujus radicatum multiplica per fractionis oblatæ numeratorem, productum partire per ejusdem denominatorem; quoti radix propositæ speciei, tam proxime minor quàm major, extrahatur integrorum more, et utrique denominator electus subscribatur: Erunt enim termini optati, ille minor, hic major, radicem quæsitam includentes.

Ut sit fractionis superioris $\frac{290}{40}$, seu $\frac{29}{4}$, extrahenda radix bipartiens: Sit denominatio optata 200 partium; harum radicatum propositæ speciei (duplicatum scilicet) est 40000; multiplica ergo 40000 per 29, numeratorem oblatum, et productum partire per 4, ejusdem denominatorem, provenient 290000 quotus; cujus radix bipartiens, nempe proxime minor, est 538, et proxime major est 539, (ex superioribus integrorum regulis) quibus subscribendo denominatorem 200 optatum, fient $\frac{538}{200}$, minor terminus, et $\frac{539}{200}$, major terminus, differentes solâ unicâ ducentesimâ unitatis, et ita arcte satis includentes $\sqcup\frac{29}{4}$, seu radicem bipartientem hujus fractionis $\frac{29}{4}$ quæsitam.

Exemplum radicis tripartientis.

Item, sit extrahenda radix tripartiens è $\frac{2}{3}$, seu potius (quia hæc radix non est numero explicabilis) quærantur ejus termini

major et minor, denominationem quam volueris, exempli gratiâ, mille partium habentes; inter quos vera tripartiens radix duarum tertiarum, seu $\llcorner \frac{2}{3}$, contineatur. Triplicatum ergo mille partium, quod est 1000000000, multiplica per 2, fient 2000000000; quæ partire per 3, et fient $666666666\frac{2}{3}$; horum radix tripartiens minor est 873, et major est 874, quibus subscribatur electus denominator 1000, fientque $\frac{873}{1000}$ minor, et $\frac{874}{1000}$ major terminus radicis tripartientis duarum tertiarum, seu $\llcorner \frac{2}{3}$ quæsitæ.

Ultimò omnium, fractiones, quarum numerator excedit denominatorem, ad integros numeros restituendæ sunt partiendo numeratorem per denominatorem, ut fuse satis, in fine Libri Primi, explicatum est.

Ut si ex præcedentibus operibus provenerit fractio $\frac{562}{18}$, continet hæc fractio integros per quos danda est ultima responsio potius quàm per fractos. Dividantur ergo 562 per 18, et fient 31 in quoto, et relinquuntur 4, quæ sunt $\frac{4}{18}$, seu $\frac{2}{9}$; unde, ex totâ reductione proveniunt $31\frac{2}{9}$, quæ sunt 31 integri, et duæ novenæ unitatis.

CAPUT XIV.

DE REGULIS PROPORTIONIS FRACTIONUM.

In regulis proportionis fractionum, observandæ sunt situs et operatio. In his, situs numeratorum idem est, qui et generaliter quantitatum capitis quinti Libri Primi; denominatorum verò contrarius; ita ut si numerator hâc lege infra lineam ceciderit, denominator invertetur,

et cadet supra; et contra, si ille supra, hic (ut par est) cadet infra.

Ut si proponatur è cisternâ aliquâ aquæ congii $\frac{3}{7}$ partes effluere in $\frac{5}{12}$ horæ; quæratur autem quanto tempore $2\frac{1}{2}$, seu $\frac{5}{2}$, congii effluent? Primò, ductâ lineâ ut à margine, prioris temporis numerator 5 supra lineam constituatur (per hanc et præceptum generale cap. 5 Lib. I.), quia $\frac{5}{12}$ horæ et horæ quæsitæ sunt cognomines, atque simul crescunt et decrescunt: Sic $\frac{5}{2}$ congii simul cum quæsitis horæ partibus crescunt et decrescunt; ideo, et illarum numerator 5 supra lineam collocatur: sed prioribus congii partibus $\frac{3}{7}$, crescentibus decrescunt temporis quæsiti partes, et decrescentibus crescunt; ideo illarum numerator 3 infra lineam statuitur. Contra verò denominatores omnes, scilicet, 7, 12, et 2, singuli singulis suis numeratoribus ex adverso latere lineæ opponuntur, ut à margine.

$$\frac{.5.5}{3.}$$

$$\frac{7.\ 5.5.}{3.12.2.}$$

Operatio sic perficitur. Multiplica superiores numeros invicem, et producetur numerator; item et inferiores invicem, et producetur denominator, præfatum numeratorem dividens, et optatum productum exhibens.

Ut præfati exempli superiores numeri 7, 5, 5, invicem multiplicati, faciunt 175 numeratorem; et inferiores 3, 12, 2, invicem multiplicati, producunt 72 denominatorem; unde $\frac{175}{72}$, sive (per partitionem) $2\frac{31}{72}$, horæ sunt tempus quæsitum, quo nimirum $2\frac{1}{2}$ congii aquæ effluent.

$$\frac{7.\ 5.5.}{3.12.2.} \quad \frac{175}{72}$$

Hîc etiam memor sis secundi compendii cap. 10. Lib. II. quotiescunque inter superiorum aliquem, et aliquem inferiorum, inciderit ratio composita quæ abbreviari possit.

Ut pro terminis superioris exempli $\frac{7}{3}$ $\frac{5}{12}$ et $\frac{5}{2}$, qui abbreviari nequiunt, si capiantur termini hi, $\frac{7}{6}$ et $\frac{10}{21}$ et $\frac{15}{8}$, qui rationes habent compositas, et abbreviari quidem possunt:

$\frac{7\,.\,10\,.\,15\,.}{6\,.\,21\,.\,8\,.}$

Ideo, collocatis ut à margine, dividantur 7 superiora, et 21 inferiora, per communem maximam mensuram 7, et fient $\frac{1}{3}$, hoc situ, $\frac{1\cdot 10\cdot 15\cdot}{6\cdot 3\cdot 8\cdot}$; deinde, simili modo abbrevientur 10 superius, et 6 inferius, et fient eorum loco 5 superius, et 3 inferius, hoc situ, $\frac{1\cdot 5\cdot 15\cdot}{3\cdot 3\cdot 8\cdot}$; abbrevientur tandem 15 superius, et alterutra 3 inferius, et fient 5 superius, et 1 inferius, hoc situ, $\frac{1\cdot 5\cdot 5\cdot}{3\cdot 1\cdot 8\cdot}$, vel hoc, $\frac{1\cdot 5\cdot 5\cdot}{1\cdot 3\cdot 8\cdot}$, vel aliter $\frac{1\cdot 5\cdot 5\cdot}{6\cdot 1\cdot 4\cdot}$; quorum multiplicatis superioribus 1, 5, 5, invicem, fient 25; deinde, similiter, inferioribus 3, 1, 8, vel 6, 1, 4, invicem, et fient 24: Itaque, tota fractio, quæstioni satisfaciens, erit $\frac{25}{24}$, seu, per abbreviationem, $1\frac{1}{24}$. Atque ita in omnibus similibus compendiose operandum est.

CAPUT XV.

DE FRACTIONIBUS PHYSICIS.

Perfecimus jam fractionum vulgarium computationes omnes; ordine sequerentur physicæ.

Physicas vocant partem aut partes integri partiti per statutum et vulgò receptum partitorem, denominatoris loco, à suis authoribus impositum.

Ut monetariis nostratibus placuit monetæ libram non in quotvis sed in 20 partes dividere, eisque denominationem solidi imponere; et solidum in 12 partes subdividere, quas denarios denominant. Sic medici, ponderis libram in partes 12, quas uncias nominant, et unciam in 8 drachmas, et drachmam in 3 scrupulos, etc., partiuntur. Chronologi, annum in 12 menses,

mensem in 30 (aut id circa) dies, diem in 24 horas, &c., distribuunt. Astronomi, gradum in 60 scrupula prima, seu minutias primas, primam in 60 secundas, secundam in 60 tertias, et ita deinceps, partiuntur.

In fractionibus physicis, hoc cum vulgaribus commune est, quòd quoties ultra suum denominatorem accreverint, superiorem locum unitate augent; et quoties major fractio à minore subducenda sit, præstanda est ei superioris loci unitas ad defectum supplendum.

Ut si fuerint 14 horæ addendæ ad 19 horas, quæ constituunt 33 horas, pro quibus non scribuntur 33 horæ, sed, sub titulo dierum et horarum, scribuntur dies 1 et 9 horæ; quia ultra 9 horas accreverunt 24 horæ in diem. Ita, pro 7 et 9 denariis addendis, non scribuntur 16 denarii, sed 4 denarii, et, pro accretis reliquis 12 denariis, 1 sub titulo solidi inscribenda. Item, pro 48 scrupulis primis addendis ad 15 scrupula prima, non scribuntur 63 scrupula prima, sed 3 tantum, et 1 gradus, pro reliquis 60 scrupulis. Haud secus, prout tam in integris quàm in fractis vulgaribus numeris, fieri solet; ut cum addenda sunt quinque centena ad octo centena, non scribuntur, nec pronunciantur, 13 centena, sed mille trecenti, hoc modo, 1300. Et in fractis vulgaribus addendis, veluti sex septenæ seu $\frac{6}{7}$, et quinque septenæ seu $\frac{5}{7}$ additæ, et debito modo abbreviatæ, non sunt dicendæ undecim septenæ, sed quatuor septenæ, et, pro reliquis septem septenis, scribenda est unitas, hoc situ, $1\frac{4}{7}$. Sic, in substractione, si fuerint substrahendæ 14 horæ, superioris exempli, ex uno die et 9 horis,—quia 14 horæ excedunt 9 horas, ideo resolvendus est dies ille unicus in 24 horas, quæ cum 9 horis efficiunt 33 horas; è quibus subductis 14 horis, relinquuntur, ut supra, 19 horæ quæsitæ. Sic, ex 1 solido et 4

denariis, si substrahendi sint 9 denarii, qui excedunt 4 denarios, et ideo 4 denariis præstandum est unum solidum, seu 12 denarii, ut fiant inde 16 denarii, è quibus jam subductis 9 denariis, remanent, ut supra, 7 denarii quæsiti. Sic, si è gradu et 3 scrupulis primis fuerint 15 scrupula prima substrahenda, reducenda sunt unus gradus et tria scrupula ad 63 scrupula prima, ut inde 15 auferri possint, et remanebunt 48 scrupula prima. Haud secus ac in vulgaribus integris et fractis, unitas præcedentis loci præstanda est sequenti loco ad defectum ejus supplendum. Ut si è mille trecentum auferantur quinque centena, ter centenis præstandum est unum millenarium, ut hinc fiant tredecim centena, è quibus jam aufer quinque centena, supersunt octo centena, ut supra. Sic, ex uno et quatuor septenis, auferendæ sunt sex septenæ; ut hoc fiat, reducenda est unitas in septem septenas, quæ præstandæ sunt quatuor septenis, ut fiant undecim septenæ; è quibus sex septenis ablatis, remanent quinque septenæ, ut supra. Vides itaque communem horum omnium concentum.

Est enim omnium, tam fractorum quàm integrorum, par ratio et respectus ad suas denominationes, tam ascendendo quàm descendendo.

Ut integrorum unitatum ad suas denas, centenas, et millenas superiores; et ad suas decimas, centesimas, et millesimas inferiores. Sic, scrupulorum primorum ad suos sexagenos (qui gradus sunt), et sexagenos gradus (qui duo signa sunt), ascendendo; et ad suas sexagesimas partes (quæ scrupula secunda sunt), et scrupulorum secundorum sexagesimas descendendo (quæ scrupula tertia sunt). Item, solidi ad sua vigecupla ascendendo (quæ libræ dicuntur), et ad suas duodecimas descendendo (quæ denarii dicuntur). Par est respectus, et computationis similitudo, in omnibus.

Hinc fit quod otiosum et superfluum foret, hisce particularem calculum texere, cum harum computatio facilius expediatur per vulgares integros et fractos numeros, quàm per suas particulares formas calculi, præsertim astronomici, ad quem etiam tabula prolixa sexagenaria requiritur, una cum molestiâ duplicium figurarum unicuique denominationi subjectarum, ubi arithmetica vulgaris requirit tantum unicam figuram unicuique loco.

> Ut patet expertis: Nam facilius et citius astronomicum thema vulgato more arithmetico computatur, et resolvitur, quàm ipso astronomico calculo; præter etiam tabulæ præfatæ, et duplicium figurarum, sub quâque denominatione, tædium. Namque vulgaris arithmetica loco primo unitatum, et loco secundo denorum, et loco tertio centenorum, et reliquorum singulorum unicam tantum figuram in singulis locis constituit. Astronomicus autem calculus in singulis suis locis, tam ascendendo quàm descendendo, sic notatis, ′ ″ ‴ ⁗, etc. binas plerumque figuras subjectas complectitur, quæ omnia confusionem pariunt.

Unde, omnibus fractionibus præter jam expositas vulgares omissis, huic Arithmeticæ Logisticæ finem imponimus.

DEO autem OPT. MAX., et suis Numeris omnibus Infinito, Immenso, et Perfecto, retribuatur omnis laus, honor, et gloria in æternum. Amen.

FINIS LIBRI SECUNDI.

LIBER TERTIUS.

DE LOGISTICA GEOMETRICA.

CAPUT I.

DE NOTATIONE ET NOMINATIONE CONCRETORUM.

PRÆCEDENTE Libro Arithmeticam docuimus, hîc ordine Logistica Geometrica sequitur.

GEOMETRICA ergo dicitur Logistica quantitatum concretarum per numeros concretos.

Concretus dicitur omnis numerus quatenus quantitatem concretam et continuam referat.

> Ut $3a$, si tres lineas digitales referat sic —— —— ——, est numerus discretus: Quum autem tridigitalem lineam concretam et continuam refert, hoc modo ——+——+——, dicitur numerus concretus, sed hoc improprie et ratione subjecti.

Proprie autem, et per se, concretos numeros dicimus radices numerorum quæ nullo numero (sive integro sive fracto) mensurari possunt.

Ut radix bipartiens, seu quadrata, septenarii major est binario, minor ternario, et nulli fracto, in universâ fractorum numerorum essentiâ, æqualis aut commensurabilis reperietur; dicitur ergo concretus numerus proprie. Sic radix tripartiens, seu cubica, denarii numeri, non est numerus discretus, nec numero commensurabilis, sed concretus; et aliæ infinitæ numerorum radices, quas vulgò surdos et irrationales vocant.

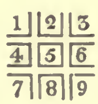
Horum concretorum ortus habetur extrahendo è numeris radices eis non insitas.

Ut cap. 4 Lib. I. et cap. 9 Lib. II. monuimus.

Unde ex diversitate radicum oriuntur diversæ notationes et nominationes concretorum.

Ut radicem bipartientem septenarii (quam vulgò radicem quadratam septenarii vocant) sic notamus, ⊔7, et pronuntiamus radicem bipartientem septem seu septenarii. Item, radicem cubicam 10, nos proferimus radicem tripartientem decem, et sic scribimus, ∟10. Item, radicem quadripartientem 11 ita notamus, ⊐11. Item, radicem quintupartientem numeri sic, □; sextupartientem sic, ⊏.

Hanc characterum radicalium varietatem cum suis indicum numeris suppeditat nobis (memoriæ gratiâ) hoc unicum schema, in suas partes distinctum.



Ut in exemplis præcedentibus, ⊔ ∟ ⊐ □ ⊏ præposita numeris, suas radices bipartientem, tripartientem, quadrupartientem, quintupartientem, sextupartientem denotabant. Sicut et ┐ radicem septupartientem; ⊓ octupartientem; ┌ noncupartientem; itemque ⌋° de-

cupartientem ; _┘| undecupartientem ; _⊔ duodecupartientem ; _└ tredecupartientem ; _┘ ⊐, vel _┤, quadrudecupartientem ; _□ quindecupartientem ; _⊏ sedecupartientem ; _┐| septemdecupartientem ; _⊓ octodecupartientem ; _┌ novemdecupartientem ; ⊔° vigecupartientem ; ⊔| 21tem ; ⊔⊔ 22tem ; ⊔└ 23tem ; ⊔⊐, vel ⊔┤ 24tem, et cætera : Itemque └₀ 30m ; ⊐° 40m ; □° 50m ; ⊏₀ 60m ; ┐°, vel ┐₀, 70m ; ⊓₀ 80m ; ┌° 90m ; _┘°° 100m : Et ita in infinitum, more figurarum arithmeticarum.

Geometrici numeri eò quod quantitatem potius nominent quàm numerent, ideo vulgò nomina dicuntur.

Nominum alia sunt unius nominis, ut uninomia ; alia plurium.

Uninomium est idem quod concretus numerus unicus, sive proprie, sive improprie dictus.

Unde sequitur quod uninomium vel est numerus unicus simplex, vel unici numeri simplicis radix aliquæ.

Ut 10 sunt numerus simplex, et à geometris usurpatur pro uninomio. Item ⊔10, └12, ⊐26, et similia sunt radices numerorum, et vere uninomia radicata sigillatim sumpta.

Cumque ita radicatum uninomium sit vel abundantis vel defectivi numeri radix, ejusque index vel par vel impar,—quadrifario hoc casu sequetur, quædam uninomia esse abundantia, quædam defectiva, quædam et abundantia et defectiva, quæ gemina dicimus ; quædam tandem nec sunt abundantia nec defectiva, quæ nugacia vocamus.

Hujus arcani magni algebraici fundamentum superius Lib. I. cap. 6, jecimus : quod (quamvis à nemine quod sciam revelatum sit) quantum tamen emolumenti adferat huic arti, et cæteris mathematicis, postea patebit.

In uninomiis abundantibus et defectivis, non multum refert an debita copula præponatur an interponatur; præstat tamen eam præponere. In uninomiis autem geminis et nugacibus, copula debita est semper interponenda.

Primi casus exemplum est └10, seu (quod per cap. 6 Lib. I. idem est) └+10, est (per cap. 6 Lib. I.) uninomium abundans. Secundi casus exemplum, └—10, est uninomium defectivum (cap. eodem). Tertii casus exemplum est └┘10, seu └┘+10 (quæ, ut supra, eadem sunt), significat tam quantitatem abundantem, quæ in se ducta facit +10, quàm defectivam quæ in se ducta facit etiam +10: Veluti, lucidioris exempli gratiâ, └┘9, seu └┘+9, est tam +3 quàm —3; ut superius Lib. I. cap. 6 demonstravimus. Ultimi casus exemplum est └┘—9, quod ex meris nugacibus est, nec quicquam significat quod vel abundet vel deficiat; nam novenarius defectivus nullam habet radicem bipartientem, ut Lib. I. cap. 6 patet.

In nugacibus summopere cavendum est ne copula — minutionis, interponenda, præponatur.

Ut si, pro └┘—9 (quæ est radix bipartiens minuti novenarii, et absurdum atque impossibile infert), sumpseris —└┘9, quæ quantitatem minutam radice bipartiente novenarii significat, longe aberrabis: Radix enim bipartiens novenarii hìc abundantis (scilicet └┘9) gemina est, scilicet +3 et —3, id est, ternarius abundans et ternarius defectivus; et ita quantitas his geminis +3 et —3, minuta gemina erit; qui itaque pro └┘—9 ponit —└┘9, pro absurdo et impossibili, et quantitate nugaci et nihil significante, profert quantitatem geminæ seu duplicis significationis; ab hoc ergo, in quo plurimi errarunt, cavendum.

In cæteris uninomiis (significativis scilicet) idem est copulam inter signum radicale et numerum interponere, sive utrique præponere : Nec in uninomiis illis valorem mutat, primo vel medio etiam loco vacuo (per cap. 6 Lib. I.) copulam $+$ inserere.

> Ut $\sqcup 9$, et $\sqcup + 9$, et $+ \sqcup 9$, et $+ \sqcup + 9$, idem prorsus significat, videlicet tam $+3$ quàm -3 ; item $\llcorner 27$, seu $\llcorner + 27$, seu $+ \llcorner 27$, seu $+ \llcorner + 27$, idem valent quod $+3$ tantummodo ; item $\llcorner - 27$, seu $+ \llcorner - 27$, seu $- \llcorner 27$, seu $- \llcorner + 27$, idem valent quod -3 tantummodo ; item in nugantibus idem est $\sqcup - 9$, et $+ \sqcup - 9$, scilicet eandem impossibilitatem implicant ; sed cave ne pro ipsis posueris $- \sqcup 9$, seu $- \sqcup + 9$, ut præcedente sectione monuimus.

Atque hæ sunt affectiones uninomiorum in se ; sequuntur uninomiorum ad invicem affectiones.

Sunt itaque uninomia bina, aut invicem commensurabilia aut incommensurabilia.

Commensurabilia sunt, quæ se habent ad invicem ut numeri discreti, seu absoluti.

Unde, omnis numerus absolutus omni numero absoluto est commensurabilis. Itemque, uninomia bina consimiliter radicata, quorum alterius numerus simplex, numerum simplicem alterius partitus, reddit numerum tali radice præditum qualem radicale signum indicat, dicuntur ad invicem commensurabilia in ratione quam indicat radix.

> Ut 5 ad 7, quia sunt numeri absoluti, seu rationales, sunt commensurabiles ; item, sint bina uninomia consimiliter radicata, $\sqcup 8$ et $\sqcup 2$, quorum numerus simplex 8, per numerum simplicem 2 partitus, reddit 4 ; habet autem quaternarius radicem signi $\sqcup$, scilicet bipartientem, estque binarius ; sunt ergo $\sqcup 8$ et $\sqcup 2$ commensurabiles invicem in ratione radicis, scilicet dupla.

Cætera omnia uninomia ad hæc irreducibilia, incommensurabilia esse constat.

Ut └─12 et └┘3, quia sunt dissimiliter radicata, sunt incommensurabilia; item, └┘6 et └┘2 (quamvis sint consimiliter radicata) sunt incommensurabilia, quia 6 per 2 partita producuntur 3, quæ carent radice signi └┘, scilicet bipartiente; at 12 et └┘4 sunt commensurabilia, quia reducta idem valent quod 12 et 2.

Et cætera.

I could find no more of this geometricall part amongst all his fragments

ALGEBRA JOANNIS NAPERI

MERCHISTONII BARONIS.

LIBER PRIMUS.

DE NOMINATA ALGEBRÆ PARTE.

CAPUT I.

DE DEFINITIONIBUS ET DIVISIONIBUS PARTIUM, ET DE VOCABULIS ARTIS.

1. ALGEBRA Scientia est de quæstionibus quanti, et quoti, solvendis tractans.

2. Estque ea duplex,—altera nominatorum, altera positivorum.

3. Nominata sunt, quæ à numeris rationalibus, aut irrationalibus, nomen habent.

4. Rationales sunt numeri absoluti, aut numeri partes; de quibus tractat etiam Arithmetica.

5. Irrationales sunt radices numerorum rationalium non habentes radices inter numeros.

6. Atque hæ (quòd quantitates sunt) in Geometriam etiam spectant.

7. Positiva Algebræ pars est, quæ quantitates et numeros latentes per suppositiones fictitias prodit; de quâ Libro II. tractabimus.

8. Priorem autem Algebræ partem, de numeris et quantitatibus nominatis, hoc Libro I. docebimus.

9. Suntque nominatorum tres species: Uninomia, plurinomia, et universalia; de quibus ordine agetur.

10. Uninomia sunt, numerus unicus simplex, aut numeri simplicis radix aliqua.

11. Atque radices numerorum diversæ sunt; diversis igitur characteribus præpositis, artis et doctrinæ gratiâ, exprimuntur; dicunturque hi characteres signa radicalia.

Ut radix cubica senarii sic scribitur, √𝔠6; item, radix quadrata quinarii sic, √𝔮5; et sic de reliquis, ut sequitur:—

√𝔮	Radix quadrata.
√𝔠	Radix cubica.
√𝔮𝔮	Radix quadrati quadrata.
√ß	Radix supersolida.
√𝔮𝔠	Radix quadrati cubica.
√ʃß	Radix secunda supersolida.
√𝔮𝔮𝔮	Radix quadrati quadrati quadrata.
√𝔠𝔠	Radix cubi cubica.

Et sic de cæteris in infinitum.

12. Radicum et radicalium quædam sunt simplices:

Ut √𝔮, √𝔠, √ß, √ʃß.

Quædam multiplices:

Ut √𝔮𝔮, √𝔮𝔠, √𝔮𝔮𝔮, √𝔠𝔠, etc.

13. Rursus, radicum et radicalium quædam sunt quadratinomiæ, quæ nominantur à quadrato, sive solo, sive secus.

Ut radix quadrata, radix quadrati quadrata, radix quadrati cubica, radix quadrati quadrati quadrata, radix quadrati supersolida, etc.

Quædam vero sunt exquadratæ, in quorum nomine non fit quadrati mentio.

Ut radix cubica, radix supersolida, radix secunda supersolida, radix cubi cubica, radix cubi supersolida, etc.

14. Uninomia bina, eodem radicali signo affecta, consimiliter radicata dicuntur; et quæ diverso, dissimiliter.

15. Omnes duo numeri rationales commensurabiles sunt; itemque, uninomia bina consimiliter radicata, quorum alterius numerus simplex, per alterius numerum simplicem divisus, reddit numerum tali radice præditum qualem radicale demonstrat, dicuntur ad invicem commensurabilia.

Ut 12 ad 2, quia sunt rationales, erunt etiam commensurabiles; itemque sint bina uninomia consimiliter radicata, √𝔮8 et √𝔮2, quorum 8 per 2 divisus reddit quaternarium præditum radice quadratâ signo 𝔮 demonstratâ, quæ est 2; sunt ergo √𝔮8 et √𝔮2 commensurabilia.

Corollarium.

16. Hinc patet, cætera uninomia ad hæc irreducibilia incommensurabilia esse.

Ut √𝔮6 et √𝔮2 sunt incommensurabiles, quia 6 divisa per 2 reddunt 3, quæ carent radice quadratâ; item, √𝔠12 et √𝔮3, quia sunt dissimiliter radicata, sunt incommensurabilia; at 12 et √𝔮4 sunt commensurabiles, quia reducti idem valent quod 12 et 2.

CAPUT II.

DE UNINOMIORUM ADDITIONE.

1. Si uninomia bina proposita commensurabilia fuerint, divide majorem numerum absolutum per minorem; quotientis extrahe (per Arithmeticam) radicem qualem radicale demonstrat; huic radici adde unitatem; productum in se duc toties quoties radicale demonstrat; inde hoc in numerum absolutum minoris uninomii duc, atque producto præpone pristinum suum radicale: fiet hoc uninomium æquale prioribus binis.

Ut sint uninomia commensurabilia addenda √ꝗ12 et √ꝗ3; divide 12 per 3, fiunt 4; ex 4 extrahe √ꝗ, fiunt 2; his adde 1, fiunt 3; quæ in se duc toties quoties signum ꝗ monstrat, videlicet quadrate, fiunt 9; hæc per 3 (numerum scilicet minoris uninomii) duc, fiunt 27; quibus præpone signum pristinum, fiunt √ꝗ27, quæ est aggregatum utriusque √ꝗ12 et √ꝗ3. Item, √𝔠24 et √𝔠3 (per cap. 1 prop. 14) sunt commensurabiles, et (per hanc) additæ faciunt √𝔠81. Item, √ꝗ$\frac{2}{3}$ ad √ꝗ2$\frac{2}{3}$ additæ producunt √ꝗ6.

2. Si uninomia proposita incommensurabilia fuerint, non aliter connectuntur quàm interposito hoc signo +, quod augmenti dicitur copula.

Ut sint addendæ √𝔠5 et √ꝗ3, fiunt √𝔠5+√ꝗ3; quæ sic pronuncianda sunt,—radix cubica quinarii aucta radice quadratâ ternarii. Item, √ꝗ6 et √ꝗ2 additæ faciunt √ꝗ6+√ꝗ2.

Corollarium.

3. Hinc patet, ex additione uninomiorum incommensurabilium oriri binomia et plurinomia abundantia,—sic dicta quòd duobus aut pluribus uninomiis, copulâ + conjunctis, constent; de quibus suo loco.

CAPUT III.

DE UNINOMIORUM SUBSTRACTIONE.

1. Si uninomia proposita commensurabilia fuerint, divide numerum uninomii, ex quo substractio fit, per numerum uninomii substrahendi; quotientis extrahe radicem qualem radicale denotat; ab hâc substrahe unitatem; reliquum in se toties duc quoties indicat radicale; productum etiam in numerum uninomii substrahendi duc, atque huic præpone signum radicale pristinum, et fiet inde residuum seu reliquum substractionis priorum uninomiorum.

Exempla.

Sit substrahenda √ȝ12 ex √ȝ27; divide 27 per 12 arithmetice, fient $\frac{9}{4}$, cujus √ȝ est $\frac{3}{2}$; hinc aufer 1, fiet $\frac{1}{2}$; quod in se quadrate duc, fiet $\frac{1}{4}$; et hanc in 12 duc, fient 3; quibus præpone radicale suum, fietque √ȝ3, pro reliquo substractionis √ȝ12 ex √ȝ27. Simili modo, √𝔠24 subducta ex √𝔠81 relinquit √𝔠3. Item, √ȝ$\frac{2}{3}$ ex √ȝ6 subducta relinquit √ȝ2$\frac{2}{3}$.

2. Si uninomia proposita incommensurabilia fuerint, ambo simul scribe postposito uninomio substrahendo, et interposito hoc signo —, quod copula minutionis nuncupatur.

Exempla.

Sit ex $\surd\mathfrak{z}3$ substrahenda $\surd\mathfrak{C}5$, remanent $\surd\mathfrak{z}3 - \surd\mathfrak{C}5$, quæ sic pronunciatur, radix quadrata ternarii minuta radice cubicâ quinarii. Item, $\surd\mathfrak{z}2$ ex $\surd\mathfrak{z}3$ relinquunt $\surd\mathfrak{z}3 - \surd\mathfrak{z}2$.

Corollarium.

3. Hinc patet, ex substractione uninomiorum incommensurabilium oriri binomiorum et plurinomiorum residua defectiva, seu apotomes; ex uninomiorum enim plurium commixtione per copulam substractionis definitur apotomes.

CAPUT IV.

DE EXTRACTIONE RADICUM EX UNINOMIIS.

1. Si radix extrahenda simplex fuerit, atque in numero absoluto uninomii latitet, eam arithmetice extrahe retento pristino radicali.

Exempla.

Sit radix quadrata extrahenda ex hoc uninomio $\surd\mathfrak{C}4$; quia in 4 latitat radix quadrata, eam extrahe retento pristino radicali, eritque $\surd\mathfrak{C}2$. Item, $\surd\mathfrak{C}$ extracta ex $\surd\mathfrak{z}\frac{54}{16}$, seu $\surd\mathfrak{z}\frac{27}{8}$, erit $\surd\mathfrak{z}\frac{3}{2}$.

2. Si vero radix simplex extrahenda in numero absoluto uninomii non deprehendatur, tunc præpone numero absoluto et radicis extrahendæ signum et pristinum signum.

Ut sit extrahenda $\surd\mathfrak{z}$ ex hâc $\surd\mathfrak{z}3$, ea erit $\surd\mathfrak{z}\mathfrak{z}3$; item, $\surd\mathfrak{C}$ hujus $\surd\mathfrak{z}5$, fit $\surd\mathfrak{z}\mathfrak{C}5$.

3. Si autem radix multiplex ex uninomio extrahenda sit, primò multiplicis unam simplicem, inde aliam ex hâc, et sic omnes sigillatim (per 1 et 2 hujus), extrahe.

Ut sit extrahenda $\surd\mathfrak{q}\mathfrak{C}$ ex hâc $\surd\text{ß}27$, extrahe hinc (per 2) $\surd\mathfrak{q}$, ea erit $\surd\mathfrak{q}\text{ß}27$; deinde ex hâc extrahe (per 1) $\surd\mathfrak{C}$, ea erit $\surd\mathfrak{q}\text{ß}3$. Item, $\surd\mathfrak{q}\mathfrak{q}$ hujus $\surd\mathfrak{C}16$ erit $\surd\mathfrak{C}2$. Item, $\surd\mathfrak{q}\mathfrak{q}$ hujus $\surd\mathfrak{C}10$ erit $\surd\mathfrak{q}\mathfrak{q}\mathfrak{C}10$.

4. In radicibus extrahendis ex fractionibus, idem est præponere radicale lineæ interpositæ, ac si id utrique numeratori scilicet et denominatori præponeres.

Ut sit ex $\frac{2}{3}$ extrahenda $\surd\mathfrak{q}\mathfrak{C}$, ea erit $\frac{\surd\mathfrak{q}\mathfrak{C}2}{\surd\mathfrak{q}\mathfrak{C}3}$, seu $\surd\mathfrak{q}\frac{\surd\mathfrak{C}2}{\surd\mathfrak{C}3}$, seu $\surd\mathfrak{C}\frac{\surd\mathfrak{q}2}{\surd\mathfrak{q}3}$, seu optime $\surd\mathfrak{q}\mathfrak{C}\frac{2}{3}$; sunt enim hæc omnia penitus eadem.

CAPUT V.

DE REDUCTIONE AD IDEM RADICALE.

Si duo uninomia fuerint dissimiliter radicata, et utriusque numerum absolutum toties in se multiplicaveris, quoties dissimile radicale socii indicat; et utrique producto, per se posito, utrumque radicale præposueris; ad idem radicale reducentur, salvo valore pristino.

Exemplum.

Sint duæ dissimiliter radicatæ, $\surd\mathfrak{C}3$ et $\surd\mathfrak{q}2$, reducendæ ad idem radicale; multiplica ergo 3 in se quadrate, et 2 in se cubice, fient ex illis 9, ex his 8, quæ præpositis utrisque radicalibus fient $\surd\mathfrak{q}\mathfrak{C}9$ et $\surd\mathfrak{q}\mathfrak{C}8$, quæ consimiliter radicatæ

sunt, retento etiam valore pristino; idem enim valet √♀ꞇ9 quòd √ꞇ3, et idem valet √♀ꞇ8 quòd √♀2, ut per cap. 4 Lib. I. patet. Item, √♀♀2 et √♀ꞇ5 sic reducuntur: multiplica 2 cubice in se, et 5 in se quadrate (in priore enim quadrato sunt similes), fient √♀♀ꞇ8 et √♀♀ꞇ25. Item, √♀6 et 2 sunt √♀6 et √♀4.

CAPUT VI.

DE MULTIPLICATIONE ET DIVISIONE UNINOMIORUM.

1. Omne uninomium, nullâ notatum copulâ, notari copulâ augmenti subintelligitur.

Ut √♀10 pro +√♀10 habetur.

2. Eadem copula per eandem multiplicata, aut divisa, producit augmenti copulam; contrariæ autem copulæ invicem multiplicatæ, aut divisæ, producunt minutionis copulam.

Exempla sunt inferius.

3. Primò ergo uninomia proposita fiant consimiliter radicata, si non per se, saltem per reductionem; deinde multiplica aut divide numerum per numerum, producti radicem qualem indicat radicale extrahe, per cap. 4; ultimò (per 2 hujus) multiplica aut divide copulas, copulamque productam præfatæ radici præpone, fietque inde multiplicationis aut divisionis productum.

Sint uninomia multiplicanda √♀12 per √♀3; duc 12 in 3, fiunt 36, quorum radix quadrata est 6, quæ sunt productum. Item, √ꞇ3 per —√ꞇ2 ducta facit —√ꞇ6 productum, per 2

hujus. Item, —√ℭ3 per √♀2 multiplicantur, prius factâ reductione ad √♀ℭ9 et √♀ℭ8, per cap. 5; inde 9 in 8 ducta faciunt 72, quorum radix ♀ℭ (per cap. 4 sect. 2) erit √♀ℭ72, cui præpone copulam —, per 2 hujus productam, fiet —√♀ℭ72.

Exempla divisionis.

Sit dividenda √♀12 per —√♀3, fiet, per præmissa, productus quotiens —2. Item, —√ℭ16 per √ℭ2 divisa fiet —2. Item, √♀ℭ72 per √♀ℭ9 divisa facit √♀ℭ8, alias √♀2; ut ex capitis 4 sectione 1 patet.

Corollarium.

4. Hinc patet uninomium quadrate, cubice, aut ad aliquem ordinem multiplicari, cum numerus ejus absolutus in se quadrate, cubice, aut ad illum ordinem multiplicatur, retento pristino radicali.

Ut √♀2 cubice ducta fit √♀8. Item, √ℭ3 quadrate ducta fit √ℭ9, etc.

Corollarium.

5. Hinc sequitur unum quodvis radicatum in se toties multiplicari, quoties suum radicale quadratinomium indicat, quum copula præcedens et radicale auferuntur; aut non quadratinomium, cum radicale tantum aufertur.

Ut —√ℭ5 in se cubice ducta facit —5. Item √♀3 in se quadrate ducta facit 3. Item, √♀ℭ6 in se quadrate ducta facit √ℭ6, cubice verò, √♀6. Simili modo fit in universalibus radicibus, et in positivis, de quibus postea.

CAPUT VII.

DE PLURINOMIIS.

1. Plurinomia sunt, quæ pluribus uninomiis copulatis constant.

2. Plurinomiorum alia abundantia dicuntur (cap. 2 sect. 3 descripta), alia defectiva, vulgò residua seu apotomes, quæ cap. 3 sect. 3 describuntur.

3. Plurinomia infima sunt ea, quorum uninomia omnia sunt quadratæ numerorum radices, cum numero vel sine numero.

Ut $\surd\mathfrak{q}3+\surd\mathfrak{q}5-\surd\mathfrak{q}2+5$, item $\surd\mathfrak{q}3+\surd\mathfrak{q}5-\surd\mathfrak{q}2$, dicuntur plurinomia infima.

4. Cætera omnia plurinomia dicuntur superiora.

5. Alia aliis magis aut minus plurinomia dicuntur, quantò plura aut pauciora fuerint uninomia.

Ut quadrinomium $\surd\mathfrak{q}3+\surd\mathfrak{q}5-\surd\mathfrak{q}2+5$ magis plurinomium hoc trinomio $\surd\mathfrak{q}3+\surd\mathfrak{q}5-\surd\mathfrak{q}2$, quod trinomium rursus magis plurinomium est quàm binomium, ut $\surd\mathfrak{q}6-\surd\mathfrak{q}5$.

6. Si dati plurinomii convertas quasdam copulas, non autem omnes, ex abundante facies suum defectivum seu apotomen; aut contra, ex defectivo suum abundans fiet.

Ut $\surd\mathfrak{q}5+\surd\mathfrak{q}3-\surd\mathfrak{q}2$ sit trinomium abundans, ejus defectivum erit $\surd\mathfrak{q}5-\surd\mathfrak{q}3-\surd\mathfrak{q}2$, vel $\surd\mathfrak{q}5-\surd\mathfrak{q}3+\surd\mathfrak{q}2$, vel $\surd\mathfrak{q}3-\surd\mathfrak{q}5+\surd\mathfrak{q}2$, vel $\surd\mathfrak{q}3-\surd\mathfrak{q}5-\surd\mathfrak{q}2$. Item, sit idem trinomium $\surd\mathfrak{q}5+\surd\mathfrak{q}3-\surd\mathfrak{q}2$ defectivum, ejus abundans erit $\surd\mathfrak{q}5+\surd\mathfrak{q}3+\surd\mathfrak{q}2$. Quo exemplo patet, idem plurinomium posse et abundans et defectivum esse, diverso tamen respectu.

7. Si plurinomii dati fuerint bina uninomia commensurabilia ejusdem copulæ, ea (per cap. 2 s. 1) adde, copulamque illam producto præpone, et fiet abbreviatio magis plurinomii in minus.

Sit trinomium hoc $\surd q12+\surd q3-2$, cujus $\surd q12$ et $\surd q3$ commensurabiles, et ejusdem copulæ, additæ facient $\surd q27$, quæ cum -2 facient abbreviationem ad minus plurinomium, viz. ad $\surd q27-2$, quod binomium est.

8. Si plurinomii dati fuerint bina uninomia commensurabilia diversarum copularum, minus à majore substrahe, (per cap. 3 s. 1) et producto præpone copulam majoris uninomii, et fiet abbreviatio magis plurinomii in minus.

Ut sit trinomium $\surd q10+\surd q2-\surd q8$, in quo $+\surd q2$ et $-\surd q8$ sunt commensurabiles, et diversis copulis notantur; substracta ergo faciunt $\surd q2$, cui copulam majoris uninomii præpone, fiet $-\surd q2$, quæ cum $\surd q10$ faciunt abbreviationem ad binomium, viz. $\surd q10-\surd q2$.

CAPUT VIII.

DE ADDITIONE PLURINOMIORUM.

1. Plurinomiorum addendorum omnia uninomia simul cum copulis suis in unicum plurinomium connecte; deinde si quæ fuerint commensurabilia, ea (per 7 et 8 præcedentis) abbreviato, et inde producitur additionis summa.

Ut sint addenda $\surd q3+\surd q8$ ad $4-\surd q2$, primò per hanc fient $\surd q3+\surd q8+4-\surd q2$; deinde, quia $+\surd q8$ et $-\surd q2$ sunt commensurabiles, ideo fiet abbreviatio (per cap. 7 sect. 8)

ad √q3+√q2+4. Item, sint addenda √q5+√q3 ad √q20—√q12, ea primò per hanc fient √q5+√q3+√q20—√q12; deinde, per dictam abbreviationem, erunt √q45—√q3. Item, √c16+√q18, ad √c2—√q2, faciunt √c54+√q8. Item, √c54+√q18—1, ad √q2+√q3, faciunt √c54+√q32+√q3—1.

Corollarium.

2. Hinc patet, in additione abundantium ad sua defectiva, particulas abundantes et defectivas sese mutuò destruere, particulas verò reliquas duplari.

Ut abundans 12+√q3 additum ad suam apotomen 12—√q3, facit 12+√q3+12—√q3, quod idem valet quod 24.

CAPUT IX.

DE SUBSTRACTIONE PLURINOMIORUM.

1. Plurinomii substrahendi converte omnes copulas, deinde hoc conversum plurinomium addatur (per caput præcedens) ad plurinomium ex quâ fieri debuit substractio, et producentur inde substractionis reliquiæ.

Ut à binomio √q45—√q3 substrahendum sit √q5+√q3, cujus converte copulas, sic, —√q5—√q3, hoc ad √q45—√q3 adde (per cap. præcedens), fiet √q20—√q12. Item, ex √c54+√q3 substrahendum sit √c2—√q2, remanebit √c16+√q18. Item, ex √c54+√q32+√q3—1 sint substrahenda √q2+√q3, remanebit √c54+√q18—1.

Corollarium.

2. Hinc patet, in substractione defectivi à suo abundante, particulas abundantes seu defectivas duplari, cæteras verò se invicem destruere.

Ut ex abundante √ℨ13+7 sit substrahendum suum defectivum √ℨ13—7, primò fiet √ℨ13+7—√ℨ13+7, inde fiet ex his 14.

CAPUT X.

DE MULTIPLICATIONE PLURINOMIORUM.

1. Singula multiplicandi uninomia duc in singula multiplicantis, per cap. 6 ; aggregatum autem (si quæ habet commensurabilia) abbrevia, per sect. 7 et 8 cap. 7.

Ut √ℨ3—√ℭ2+6 sit multiplicandum, √ℨ5—7 sit multiplicans ; √ℨ15—√ℨℭ500+√ℨ180—√ℨ147+√ℭ686—42 erit productum, undique incommensurabile et ideo inabbreviabile. Item, sit √ℨ8+√ℨ3—5 multiplicandum, √ℨ12—√ℨ2 multiplicans ; √ℨ96+√ℨ36 (alias 6) —√ℨ300—√ℨ16 (alias —4) —√ℨ6+√ℨ50, quod productum (per sect. 7 et 8 cap. 7) abbreviatum facit √ℨ54+2—√ℨ300+√ℨ50.

2. In multiplicando abundante per suum defectivum, sufficit partem abundantem per partem defectivam, atque partem utrique communem in se multiplicare ; reliquiæ enim transversæ multiplicationes se invicem destruunt.

Ut √ℨ7+√ℨ5 abundans multiplicandum per √ℨ7—√ℨ5, suum defectivum, fiet 7—5 (alias 2) pro totali producto ; etenim transversæ multiplicationes, √ℨ7 per —√ℨ5, et √ℨ7 per

+√q5, sunt —√q35 et +√q35, quæ se invicem destruunt, utque igitur inutiles.

Corollarium.

3. Si infimum plurinomium abundans in suum defectivum ducatur, producit minus plurinomium infimum.

Ut multiplicetur hoc trinomium infimum abundans √q11—√q3+√q2, per suorum defectivorum aliquod, viz. per √q11—√q3—√q2, producetur inde binomium hoc 12—√q132, quod etiam per suum abundans 12+√q132 multiplicatum facit uninomium imo numerum, viz. 12.

Corollarium.

4. Si infimum binomium abundans in suum defectivum ducatur, producitur numerus.

Ut, in jam dicto, si binomium abundans 12+√q132 duxeris (per 2 hujus) in suum defectivum 12—√q132, producitur numerus, viz. 12.

Annotandum est, quòd binomium irrationale per tale plurinomium multiplicari possit, ut inde numerus rationalis proveniat, hoc modo: Duo nomina cubica duc in se et invicem, et fiet trinomium abundans, ex abundante binomio, aut defectivum ex defectivo; hoc trinomium, si abundans sit, per binomium defectivum ducatur, aut, si defectivum, per abundans, et proveniet numerus simplex. Aliter: Per prop. 2 lib. viii. Euclid., quære tres quantitates in eâ ratione quam habent invicem nomina cubica; aut quatuor quantitates in eâ ratione quam habent binomia biquadrata; aut quinque pro supersolidis; et deinde duc ut supra.

Exemplum.

Ex binomio abundante $\sqrt{\mathfrak{C}}6+\sqrt{\mathfrak{C}}4$ fac trinomium $\sqrt{\mathfrak{C}}36+\sqrt{\mathfrak{C}}24+\sqrt{\mathfrak{C}}16$, quod per defectivum $\sqrt{\mathfrak{C}}6-\sqrt{\mathfrak{C}}4$ duc, fiunt 2. Item, ex defectivo $\sqrt{\mathfrak{C}}6-\sqrt{\mathfrak{C}}4$ fiat trinomium defectivum $\sqrt{\mathfrak{C}}36-\sqrt{\mathfrak{C}}24+\sqrt{\mathfrak{C}}16$, quod per binomium abundans $\sqrt{\mathfrak{C}}6+\sqrt{\mathfrak{C}}4$ duc, fient 10.

Aliud exemplum.

Ex $\sqrt{\mathfrak{z}\mathfrak{z}}3-\sqrt{\mathfrak{z}\mathfrak{z}}2$, fit quadrinomium $\sqrt{\mathfrak{z}\mathfrak{z}}27-\sqrt{\mathfrak{z}\mathfrak{z}}18+\sqrt{\mathfrak{z}\mathfrak{z}}12-\sqrt{\mathfrak{z}\mathfrak{z}}8$, quod per $\sqrt{\mathfrak{z}\mathfrak{z}}3+\sqrt{\mathfrak{z}\mathfrak{z}}2$ duc, fiet 1.

CAPUT XI.

DE DIVISIONE PLURINOMIORUM.

1. Si divisor fuerit uninomium, divide per illud singula dividendi uninomia, per cap. 6, et quotientis uninomia cum copulis productis connecte.

Ut sit dividendum $\sqrt{\mathfrak{z}}12+\sqrt{\mathfrak{z}}8$ per $\sqrt{\mathfrak{z}}2$, fient $\sqrt{\mathfrak{z}}6+2$. Item, sit dividendum $\sqrt{\mathfrak{z}}36300+\sqrt{\mathfrak{z}}7200-\sqrt{\mathfrak{z}}10800+\sqrt{\mathfrak{z}}6600+\sqrt{\mathfrak{z}}9900$ per 12, fiet quotiens $\sqrt{\mathfrak{z}}\frac{3025}{12}+\sqrt{\mathfrak{z}}50-\sqrt{\mathfrak{z}}75+\sqrt{\mathfrak{z}}\frac{275}{6}+\sqrt{\mathfrak{z}}\frac{275}{4}$.

2. Si plurinomium fuerit divisor, illudque infimum; ex hoc plurinomio fac (per sect. 3 et 4 capitis 10) numerum simplicem; perque eadem plurinomia per quæ multiplicaveras divisorem multiplicabis etiam dividendum; productum per dictum numerum simplicem divide, et reddetur inde quotiens prioris divisoris et dividendi.

Sint 5 dividendæ per trinomium hoc infimum $\surd q\,11 - \surd q\,3 - \surd q\,2$ quod si prius in $\surd q\,11 - \surd q\,3 + \surd q\,2$ duxeris, fiet inde $12 - \surd q\,132$; deinde hoc in $12 + \surd q\,132$ duxeris, fient (per 3 et 4 capitis 10) 12; deinde per idem trinomium $\surd q\,11 - \surd q\,3 + \surd q\,2$ multiplica dividendum, viz. 5, fient $\surd q\,275 - \surd q\,75 + \surd q\,50$; hoc rursus multiplica per præfatum binomium $12 + \surd q\,132$, et fit $\surd q\,36300 + \surd q\,7200 - \surd q\,10800 + \surd q\,6600 + \surd q\,9900$ pro novo dividendo, quo, per dicta 12 diviso, provenit quotiens $\surd q\,\frac{3025}{12} + \surd q\,50 - \surd q\,75 + \surd q\,\frac{275}{6} + \surd q\,\frac{275}{4}$ congruens præfatis 5 divisis per $\surd q\,11 - \surd q\,3 - \surd q\,2$.

Hæc sunt emendanda; nam per $6 + \surd c\,2$ fieri potest divisio, ut per omne binomium, ex fine præcedentis capitis.

3. Si divisor fuerit ex superioribus plurinomiis, vix unquam dividet integrum dividendum sine reliquiis; atque igitur inter superscriptum dividendum et subscriptum divisorem linea ducatur more fractionum arithmetices.

Ut sint $10 - \surd q\,3$ dividenda per $6 + \surd c\,2$, non aliter fiet quam interlineali divisione hoc situ $\frac{10 - \surd q\,3}{6 + \surd c\,2}$, quæ sic pronuntiantur, $10 - \surd q\,3$ divisa per $6 + \surd c\,2$.

Corollarium.

4. Hinc patet, ex divisione per superiora plurinomia oriri plurinomia irrationalia fracta.

CAPUT XII.

DE EXTRACTIONE RADICUM EX PLURINOMIIS.

1. Plurinomiorum quædam radices perspicuæ sunt, quædam obscuræ. Perspicuas dicimus, quæ non sunt magis plurinomia quàm ea quorum sunt radices.

2. Obscuras autem radices appellamus, quæ plurimis uninomiis et radicibus plurinomiorum confuse plerumque scatent.

3. Si binomii infimi oblati fuerit extrahenda radix quadrata; ex differentiâ quadratorum utriusque uninomii radicem quadratam extrahe, quam ad majus uninomium adde et ab eodem substrahe, si scilicet commensurabilia sunt (alioquin enim erit radix quæsita obscura), et ab horum dimidiis educ radices quadratas (per cap. 4), has binas radices connecte copulâ quâ prius binomium, et erit hoc binomium radix quadrata perspicua prioris binomii.

Exemplum.

Sit extrahenda radix quadrata hujus binomii defectivi 3—√♀5; quadrata uninomiorum sunt 9 et 5, quorum differentia est 4, radix quadrata hujus differentiæ est 2, commensurabiles ad 3; ea igitur 3 et 2 adde, fient 5; et etiam substrahe 2 ex 3, restat 1; ex dimidiis 5 et 1 educ radices quadratas, fient √♀$\frac{5}{2}$ et √♀$\frac{1}{2}$, quas connecte copulâ pristinâ, fientque √♀$\frac{5}{2}$—√♀$\frac{1}{2}$ radix quadrata hujus binomii 3—√♀5. Item, sit extrahenda radix quadrata ex √♀48—6; radix quadrata differentiæ quadratorum est √♀12, quæ addita et substracta ad et à √♀48, facit √♀108, et √♀12, quarum radices dimidiorum copulatæ faciunt

√ꝗꝗ27—√ꝗꝗ3 quæsitam. Item, √ꝗ24+√ꝗ18 habet radicem quadratam perspicuam hanc √ꝗꝗ$\frac{27}{2}$+√ꝗꝗ$\frac{3}{2}$.

4. Cæterorum omnium plurinomiorum radices qualescunque pro obscuris habentur.

Exemplum.

√ꝗ48+√ꝗ28 caret radice perspicuâ, quia √ꝗ differentiæ quadratorum, quæ est √ꝗ20, non est commensurabilis ad √ꝗ48, cum (per præmissam 3) deberet esse commensurabilis. Item, radix quadrata vel cubica hujus √ꝣ3+1 obscura est; et sic de omnibus, præter binomia infima jam dicta.

5. Radices autem obscuræ non aliter extrahuntur quàm præponendo signum radicale radicis cum periodo ante plurinomium oblatum, idque radicale, cum periodo sequente, universalis radicis signum dicitur; indicat enim universi plurinomii sequentis radicem.

Ut, sit extrahenda radix quadrata hujus √ꝗ48+√ꝗ28; præpone huic binomio hoc radicale √ꝗ cum periodo hâc, fietque inde √ꝗ. √ꝗ48+√ꝗ28, quæ sic pronuntiantur, radix quadrata universalis radicis quadratæ 48 auctæ radice quadratâ 28; significatur enim √ꝗ48 jungi cum radice quadratâ 28 in unam summam; ejusque totalis summæ radicem quadratam capiendam. Item, sit extrahenda radix cubica hujus √ꝣ3+√ꝗ2—1, ea erit √ꝣ. √ꝣ3+√ꝗ2—1.

Corollarium.

6. Hinc patet, ex radicum obscurarum extractione, radices universales oriri.

CAPUT XIII.

DE FRACTIONIBUS IRRATIONALIBUS.

1. Quæ in fractionibus rationalibus fieri præcipit Arithmetica, hæc in irrationalibus fractionibus per Algebram perfice.

In fractionibus autem irrationalibus plurinomiis, operamur per Arithmeticam, quatenus sunt fractiones; et per Algebram, quatenus plurinomiæ et irrationales sunt.

Ut, sint $\frac{\surd\mathfrak{z}3+2}{\surd\mathfrak{c}3}$ dividenda per $\frac{\surd\mathfrak{z}5}{\surd\mathfrak{c}2-1}$, quod, in rationalibus per Arithmeticam, fieret per transversam multiplicationem utriusque numeratoris per utriusque denominatorem seorsum; hæc ergo multiplicatio per Algebram fiat, et producetur $\frac{\surd\mathfrak{zc}108+\surd\mathfrak{c}16-\surd\mathfrak{z}3-2}{\surd\mathfrak{zc}1125}$, quotiens optatæ divisionis.

Item, sint addenda $\frac{\surd\mathfrak{z}3+2}{\surd\mathfrak{c}3}$ ad $\frac{\surd\mathfrak{z}5}{\surd\mathfrak{c}2-1}$, quæ et multiplicari ex transverso, et recto denominatores, ut sint ejusdem denominationis, præcipit Arithmetica; algebraice ergo sic multiplicentur, et fient unius primò denominationis sic, $\frac{\surd\mathfrak{zc}108+\surd\mathfrak{c}16-\surd\mathfrak{z}3-2}{\surd\mathfrak{c}6-\surd\mathfrak{c}3}$ pro uno, et $\frac{\surd\mathfrak{zc}1125}{\surd\mathfrak{c}6-\surd\mathfrak{c}3}$ pro altero; deinde per Algebram, jubente Arithmeticâ, adde numeratores retento communi illo denominatore, fietque $\frac{\surd\mathfrak{zc}1125+\surd\mathfrak{zc}108+\surd\mathfrak{c}16-\surd\mathfrak{z}3-2}{\surd\mathfrak{c}6-\surd\mathfrak{c}3}$, productum additionis.

CAPUT XIV.

DE UNIVERSALIUM RADICUM ADDITIONE ET SUBSTRACTIONE.

1. Universales adduntur copulâ augmenti, et substrahuntur copulâ minutionis interpositis.

Sint addenda $\surd\mathcal{Q}10+\surd\mathcal{Q}2$ ad $\surd\mathcal{Q}.\,8-\surd\mathcal{Q}3$, interpone copulam $+$, fietque $\surd\mathcal{Q}.\,10+\surd\mathcal{Q}2+\surd\mathcal{Q}.\,8-\surd\mathcal{Q}3$. Item, substrahatur $\surd\mathcal{Q}.8-\surd\mathcal{Q}3$ ex $\surd\mathcal{Q}.\,10+\surd\mathcal{Q}2$, interpone copulam $-$, fietque $\surd\mathcal{Q}.\,10+\surd\mathcal{Q}2-\surd\mathcal{Q}.\,8-\surd\mathcal{Q}3$.

2. Si radix quadrata universalis, binomii infimi defectivi, ad radicem quadratam universalem sui abundantis addatur, aut à radice quadratâ universali sui abundanti auferatur, per copulas $+$ et $-$, productum (per cap. 16 sect. 5 sequens) erit abbreviandum.

Ut ex additione $\surd\mathcal{Q}.10-\surd\mathcal{Q}2$ ad $\surd\mathcal{Q}.10+\surd\mathcal{Q}2$ producetur, $\surd\mathcal{Q}.10+\surd\mathcal{Q}2+\surd\mathcal{Q}.10-\surd\mathcal{Q}2$; quod per cap. 16 sect. 5 abbreviabile est.

CAPUT XV.

DE UNIVERSALIUM DIVERSORUM AD IDEM SIGNUM REDUCTIONE.

1. Multiplica utrumque plurinomium universalis toties in se quoties dissimile universale socii indicat, et productum universali utriusque signabis.

Ut sint reducenda $\surd\mathcal{Q}\mathcal{Q}.\,2-\surd\mathcal{C}3$, et $\surd\mathcal{Q}\mathcal{C}.\,7+\surd\mathcal{Q}2$; duc plurinomium $2-\surd\mathcal{C}3$ in se cubice, et $7+\surd\mathcal{Q}2$ in se quadrate, fient $5+\surd\mathcal{C}1944-\surd\mathcal{C}5184$ et $51+\surd\mathcal{Q}392$, quibus præpone

$\surd\mathfrak{q}$ commune, una cum $\mathfrak{q}$ et $\mathfrak{c}$ dissimilibus, fientque $\surd\mathfrak{q}\mathfrak{q}\mathfrak{c}.\ 5+\surd\mathfrak{c}1944-\surd\mathfrak{c}5184$ et $\surd\mathfrak{q}\mathfrak{q}\mathfrak{c}.\ 51+\surd\mathfrak{q}392$ reducta ad idem universale, viz. $\surd\mathfrak{q}\mathfrak{q}\mathfrak{c}$.

2. Eâdem ratione, universalia cum particularibus ad idem radicale reducuntur.

Ut 3. et $\surd\mathfrak{q}.18+\surd\mathfrak{q}243$, fiunt $\surd\mathfrak{q}.9$ et $\surd\mathfrak{q}.18+\surd\mathfrak{q}243$. Item, $\surd\mathfrak{q}.\ 13+\surd\mathfrak{q}20$ et $2+\surd\mathfrak{q}3$, fiunt $\surd\mathfrak{q}.\ 13+\surd\mathfrak{q}20$, et $\surd\mathfrak{q}.7+\surd\mathfrak{q}48$.

Corollarium.

3. Hinc patet, uninomium signari universali quod particulari radicali idem est.

Ut $\surd\mathfrak{q}.\ 9$ et $\surd\mathfrak{q}9$ eædem sunt. Item, $\surd\mathfrak{c}5$ et $\surd\mathfrak{c}.\ 5$; hæc dum puncto notantur universalia dicuntur.

CAPUT XVI.

DE MULTIPLICATIONE ET DIVISIONE UNIVERSALIUM.

1. Si universalis per universalem multiplicanda aut dividenda fuerit, primò fiant (per cap. 15) ejusdem signi universalis.

2. Deinde, deletis (saltem mente) signis universalibus, fiat more uninomiorum et plurinomiorum multiplicatio et divisio.

3. Ultimò, præpone producto, vel quotienti, signum universale pristinum, cum copulâ (per cap. 6 sect. 2) debitâ præcedente.

Sint ejusdem signi universalis $\surd\mathfrak{c}.\ 5+\surd\mathfrak{q}2$ et $\surd\mathfrak{c}.\ 4-\surd\mathfrak{q}3$ ad invicem multiplicanda: Duc ergo $5+\surd\mathfrak{q}2$ per $4-\surd\mathfrak{q}3$, producentur $20+\surd\mathfrak{q}32-\surd\mathfrak{q}75-\surd\mathfrak{q}6$, quibus præpone $\surd\mathfrak{c}.$ vel

+√c. fient √c. 20+√q32—√q75—√q6. Item, √q. 4+ √c2 sint ducenda in 3 seu in √q9, seu (per cap. 15 sect. 3) in √q. 9: Duc ergo 4+√c2 in 9, fient 36+√c1458, per cap. 6 et cap. 10. His præpone √q. vel +√q. fient √q. 36+√c1458. Item, √q. 10+√q2 ductum in —√q. 10—√q2 facit —√q. 98, seu —√q98.

4. Si universalis quadrata, copulâ + prænotata, sit ducenda in eandem universalem, copulâ — prænotatam; prænotatam copulam et radicale universale dele, et reliquiarum copulas in contrarias converte, et orietur inde multiplicationis productum.

Ut +√q.2—√q3, per —√q.2—√q3 ducta, facit +√q3—2.

5. Verùm, si universale quadratum in se ducendum sit, deleto signo universali cum copulâ prænotatâ, orietur multiplicationis productum.

Ut √q. 10+√q2 multiplicetur in se, producetur 10+√q2.

6. Si autem plures per plures universales multiplicandæ fuerint, aut dividendæ, quod producitur ex unicâ per unicam, totum, habebit illud unicum signum universale præpositum.

Ut √c. 10+√q5+√q. 8—√q3 multiplicentur per √q. 3+ √q6—√q. 4—√q7, hoc modo: Reduc √c. 10+√q5 cum √q. 3+√q6 ad idem universale, fiet illa, √qc. 105+√q2000, hæc autem, √qc.81+√q6534; has invicem duc, fientque (per hoc caput,) √qc.8505+√q13068000+√q13122000+√q72037350; quæ producuntur ex unicâ universali in unicam ductâ. Habet ergo hoc totum productum commune illud universale signum √qc ei præpositum. Simili modo duc 3+√q6 per 8—√q3, eique suum universale, viz. +√q. præpone, fiet +√q.24 —√q27+√q384—√q18, pro secundâ parte producti. Tertiò,

reduc $-\sqrt{q}.4-\sqrt{q}7$ cum $\sqrt{c}.10+\sqrt{q}5$ ad idem universale, fient $-\sqrt{qc}.148-\sqrt{q}21175$, et $\sqrt{qc}.105+\sqrt{q}2000$; has invicem duc, fientque $-\sqrt{qc}.15540-\sqrt{q}233454375+\sqrt{q}43808000-\sqrt{q}42350000$, pro tertiâ parte producti. Quartò, duc $8-\sqrt{q}3$ per $4-\sqrt{q}7$, et producto præpone $-\sqrt{q}$. fiet $\sqrt{q}.32-\sqrt{q}48-\sqrt{q}448+\sqrt{q}21$, pro quartâ parte producti. Quarum quatuor partium quælibet fit ex ductu unicæ tantum universalis in unicam; quare quælibet comprehenditur sub unico signo universali, fitque totum productum, $\sqrt{qc}.8505+\sqrt{q}13068000+\sqrt{q}13122000+\sqrt{q}72037350+\sqrt{q}.24-\sqrt{q}27+\sqrt{q}384-\sqrt{q}18-\sqrt{qc}.15540-\sqrt{q}233454375+\sqrt{q}43808000-\sqrt{q}42350000-\sqrt{q}.32-\sqrt{q}48-\sqrt{q}448+\sqrt{q}21$.

7. Unde fit quod radice quadratâ universali binomii infimi abundantis auctâ aut minutâ radice quadratâ sui defectivi, producto in se ducto præposueris $\sqrt{q}$., fiet inde ejusdem valoris plurinomium minus et abbreviatum.

Ut si $\sqrt{q}.10+\sqrt{q}2+\sqrt{q}.10-\sqrt{q}2$ in se duxeris, et $\sqrt{q}$. præposueris, fiet $\sqrt{q}.20+\sqrt{q}392$ æquale ad $\sqrt{q}.10+\sqrt{q}2+\sqrt{q}.10-\sqrt{q}2$ eoque brevius. Pari ratione ex $\sqrt{q}.10+\sqrt{q}2-\sqrt{q}.10-\sqrt{q}2$, fit $\sqrt{q}.20-\sqrt{q}392$.

8. Exempla divisionis universalium sunt eadem plurinomia quæ cap. 11 præcedente scribuntur; si modo eorum divisoribus dividendis et quotientibus signum universale præposueris.

Ut ex hâc et cap. 11 sect. 1, $\sqrt{c}.\sqrt{q}12+\sqrt{q}8$ per $\sqrt{qc}2$, seu quod idem est per $\sqrt{c}.\sqrt{q}2$ divisa, reddunt quotientem $\sqrt{c}.\sqrt{q}6+2$. Item, ex hâc et cap. 11 sect. 2, $\sqrt{q}.5$ divisa per $\sqrt{q}.\sqrt{q}11-\sqrt{q}3-\sqrt{q}2$ reddit quotientem $\sqrt{q}.\sqrt{q}\frac{3025}{12}+\sqrt{q}50-\sqrt{q}75+\sqrt{q}\frac{275}{6}+\sqrt{q}\frac{275}{4}$. Et sic de cæteris.

CAPUT XVII.

DE RADICUM UNIVERSALIUM EXTRACTIONE.

1. PLURINOMII oblati radicem sine respectu universalis signi per cap. 12. extrahe; et huic radici præpone pristinum suum signum universalitatis.

Ut sit extrahenda radix quadrata ex √ℭ.3—√ꝗ5, ea (per cap. 12 sect. 3) erit √ꝗ$\frac{5}{2}$—√ꝗ$\frac{1}{2}$, cui per hanc præpone suum universale √ℭ. fietque radix quæsita √ℭ.√ꝗ$\frac{5}{2}$—√ꝗ$\frac{1}{2}$. Item, radix cubica hujus √ꝗ.√ℭ3+√ꝗ2 erit (per hanc et cap. 12, sect. 5) √ꝗℭ.√ℭ3+√ꝗ2. Item, radix quadrati cubica hujus √ß.7—√ꝗ48, primò ejus radix quadrata erit √ſs.2—√ꝗ3; deinde hujus radix cubica erit √ℭß.2—√ꝗ3, pro radice quæsitâ.

2. Si ex pluribus universalibus copulatis, aut ex universalibus copulatis cum uninomiis radicem extraxeris, ea dicetur universalium universale; totique debet signum universale radicis extrahendæ præponi, lineaque per totum duci.

Ut sit extrahenda radix quadrata hujus 5+√ℭ2—√ꝗ.3—√ꝗ2, ea extrahitur præponendo signum universale radicis, una cum lineâ ductâ hoc modo √ꝗ. $\underline{\text{5+√ℭ2—√ꝗ.3—√ꝗ2.}}$

Corollarium.

3. Hinc sequitur in universalibus effectum universalis signi tantum extendi quantum linea protracta; et si nulla ducatur linea effectus universalis signi in sequens universale signum desinit ab eâque intercipitur.

Ut per $\surd q.\underline{60+\surd q16-\surd q.\ 6-\surd q4}$ significatur totius hujus $60+\surd q16-\surd q.6-\surd q4$ radicem quadratam capiendam, eaque est $\surd q62$; at si abesset linea, hoc modo, $\surd q.60+\surd q16-\surd q.\ 6-\surd q4$, tunc prioris $\surd q$. effectus et vis per $60+\surd q16$ tantum extenditur, et posterioris $\surd q$. vis per reliquum, viz. per $6-\surd q4$ extenditur. Idem ergo valet $\surd q.\underline{60+\surd q16-\surd q.6-\surd q4}$, quod $\surd q62$; atqui $\surd q.60+\surd q16-\surd q.6-\surd q4$ idem est quod 6: Et similiter in similibus.

Hæc de irrationalibus dicta sufficiunt, licet et aliæ sint irrationalium species: Ut enim per extractionem radicum ex numeris non habentibus radices oriuntur uninomia (quæ primâ parte hujus docuimus), et ex additione et substractione uninomiorum incommensurabilium oriuntur plurinomia (de quibus secundâ parte hujus tractavimus), et per extractionem radicum obscurarum ex plurinomiis oriuntur universalia (de quibus hâc tertiâ et ultimâ parte hujus tractavimus). Sic etiam ex universalibus oriuntur universalium universalia, et ex his rursus alia ad infinitum universalissima: Quorum artem si aliquando in usum cadat, quod rarissime accidit, facillime ex præcedentibus colliges.

FINIS PRIMI LIBRI.

LIBER SECUNDUS.

DE POSITIVA SIVE COSSICA ALGEBRÆ PARTE.

CAPUT I.

DE DEFINITIONIBUS ET DIVISIONIBUS PARTIUM, ET DE VOCABULIS ARTIS.

1. Positivam Algebræ partem, per suppositiones fictas, veram quantitatem verumque numerum quæsitum patefacere diximus, Lib. I. cap. 1, sect. 7.

2. Positiones etiam, sive suppositiones, sunt notulæ quædam fictæ unitate notatæ, quas loco ac vice quantitatum ac numerorum ignotorum addimus, substrahimus, multiplicamus aut dividimus.

3. Positiones autem, et positionum notulæ, tot sunt diversæ et dissimiles quot diversos, dissimiles, ignotosque numeros aut quantitates complectitur quæstio.

> Quarum, exempli gratiâ, figuræ et nomina sunt 1℞, quæ una prima positio dicitur, $1a$, quæ unum a, sive una secunda positio dicitur; $1b$, unum b, sive una tertia positio; $1c$, unum c, sive una quarta positio; et sic per alphabetum.

4. Hæ positionum notulæ (eo quod pro omnis rei numero et mensurâ incognito ponuntur) vulgari nomine Res dicuntur, suntque primæ ordine.

5. Quadratum est productum ortum ex harum rerum aliquâ in se ductâ, estque secundum ordine.

Ut 1℞ in se ducta facit unum primum quadratum, quod sic scribitur 1ℨ. Item, 1*b* in se ductum facit 1*b*ℨ, quod unum *b* quadratum dicitur. Item, 1*a* per 1*a* ductum facit 1*a*ℨ, quod unum *a* quadratum dicitur; et sic de cæteris.

6. Cubus est qui ex ductu rei cujusvis in suum quadratum oritur; estque ordine tertius.

Ut 1℞ ducta in 1ℨ facit unum cubum, qui sic scribitur 1ℭ. Item, 1*a* per 1*a*ℨ ductum facit 1*a*ℭ, qui pronuntiatur sic, unus *a* cubus. Item, 1*b* per 1*b*ℨ ductum facit 1*b*ℭ, etc.

7. Quadrati quadratum est quod ex ductu rei cujusvis in suum cubum provenit; estque ordine quartum.

Ut 1℞ ducta in 1ℭ producit unum quadrati quadratum, quod sic scribitur 1ℨℨ. Item, 1*a* per 1*a*ℭ facit 1*a*ℨℨ, quod sic pronuntiatur, unum *a* quadrati quadratum. Sic 1*b*ℨℨ, 1*c*ℨℨ, etc.

8. Supersolidus est qui ex ductu rei cujusvis in suum quadrati quadratum provenit; estque ordine quintus.

Ut 1℞ ducta in 1ℨℨ facit 1*ß*, scilicet unum supersolidum. Item, 1*a* per 1*a*ℨℨ facit 1*aß*, quod pronuntiatur unus *a* supersolidus. Sic de 1*bß*, et 1*cß*, etc.

Corollarium.

9. Hinc patet alios ex aliis oriri ordines in infinitum progredientes.

Ut 1℞ ducta per 1ß facit 1𝔷𝔠, qui ordine sextus est. Item, 1℞ per 1𝔷𝔠 facit 1ſß, qui secundus supersolidus dicitur, estque ordine septimus. Cætera ex tabellâ subsequenti contemplari licebit, in quâ supponimus exempli gratiâ 1℞ valere 3, 1*a* valere 2, et 1*b* valere 4, quibus datis cæterorum ordinum valores necessario sequentur, ut inferius :

Numeri ordinum 0	Characteres et exempla ordinum primæ positionis.		Characteres et exempla ordinum secundæ positionis.		Characteres et exempla ordinum tertiæ positionis.		&c.
1	1 ℞	3	1 *a*	2	1 *b*	4	
2	1 𝔷	9	1 *a* 𝔷	4	1 *b* 𝔷	16	
3	1 𝔠	27	1 *a* 𝔠	8	1 *b* 𝔠	64	
4	1 𝔷𝔷	81	1 *a* 𝔷𝔷	16	1 *b* 𝔷𝔷	256	
5	1 ß	243	1 *a*ß	32	1 *b* ß	1024	
6	1 𝔷𝔠	729	1 *a* 𝔷𝔠	64	1 *b* 𝔷𝔠	4096	
7	1 ſß	2187	1 *a* ſß	128	1 *b* ſß	16384	&c.
8	1 𝔷𝔷𝔷	6561	1 *a* 𝔷𝔷𝔷	256	1 *b* 𝔷𝔷𝔷	65536	
9	1 𝔠𝔠	19683	1 *a* 𝔠𝔠	512	1 *b* 𝔠𝔠	262144	
10	1 𝔷ß	59049	1 *a* 𝔷ß	1024	1 *b* 𝔷ß	1048576	
11	1 ſſß	177147	1 *a* ſſß	2048	1 *b* ſſß	4194304	
12	1 𝔷𝔷𝔠	531441	1 *a* 𝔷𝔷𝔠	4096	1 *b* 𝔷𝔷𝔠	16777216	
13	1 ſſſß	1594323	1 *a* ſſſß	8192	1 *b* ſſſß	67108864	
&c.	&c.		&c.		&c.		

10. Positivi dicuntur numeri quicunque rationales, vel irrationales, signis positivorum ordinum notantur.

Ut 6℞, vel 5*a*, vel 7*b*ꞇ, vel √Ϟ6*b*, vel √ꞇ7*a*Ϟ, positivi numeri dicuntur. Interdum etiam nomen positivi pro numero quovis capitur.

11. Simplex dicitur quivis numerus positivus unicus solus, aut solitarie sumptus.

Ut 6*a* est simplex. Item, √Ϟ3ꞇ. Item, √Ϟ1*ab*.

12. Compositus dicitur qui ex pluribus simplicibus qui signis pluris vel minoris copulantur constat.

Ut 6*a*+√Ϟ3ꞇ. Item, 5℞—2Ϟ. Item, √Ϟ30ꞇ+3*a*—4℞*b*.

13. Purus dicitur simplex qui, post unicum uninomium habet unius tantum positionis signum conscriptum.

Ut 5*a*Ϟ. Item, 3ꞇ. Item, √Ϟ2*c*ꞇ, etc., puri dicuntur.

14. Mistus dicitur simplex qui post unicum uninomium, habet diversarum positionum signa conscripta.

Ut 5Ϟ*ac*, 2℞*ac*, √Ϟ1*ab*, √ꞇ1*a*Ϟ*bβc*, et similes infiniti, misti dicuntur; de quorum origine inferius cap 5, sect. 2 et 3, tractabitur.

15. Simplices rationales sunt qui numeros rationales habent præpositos signis positionis et ordinis. Irrationales autem qui irrationales numeros præpositos habent.

Ut 6*a*, item 5Ϟ*a*Ϟ, item 2℞, rationales sunt; atqui √Ϟ6*a*, item √ꞇ5Ϟ*b*ꞇ, item √Ϟꞇ7*bβ*, sunt irrationales.

16. Similiter radicati dicuntur simplices quorum signa radicalia aut

nulla aut similia sunt; dissimiliter autem contra quorum radicalia dissimilia sunt.

Ut 2♀ et 3*a*, item √♀3℞ et √♀5ℭ, item √ℭ6 et √ℭ2*ab*, sunt similiter radicati; atqui √♀2ℭ et √ℭ3ℭ, item √ß1*a* et 5℞*b*, etc. sunt dissimiliter radicati.

17. Ejusdem positionis sunt bini simplices notati characteribus omnimode ejusdem positionis, licet non ejusdem ordinis.

Ut 2℞ et √♀5ß, item 3℞*a*♀ et √ℭ2♀*a*ℭ, etc. sunt ejusdem positionis; at 2℞ et √♀1*a*, item 3℞*a*♀ et 2℞, sunt diversarum positionum.

18. Ejusdem ordinis actu dicuntur simplices similiter radicati, quorum signa etiam ordinis eadem sint, licet non sint ejusdem positionis.

Ut 3*a* et 2*b*, item √♀2ℭ et √♀5ℭ, item 2℞*a* et 5℞*b*, item √♀2℞*a* et √♀3*bc*, sunt ejusdem ordinis actu; at 2℞*a*♀ et 3*b*, item √♀2℞*a*♀, et √♀3*b*, item √♀2℞*a* et √♀3℞*a*♀, sicut et alia ejus generis sunt diversorum ordinum. Quæ vero sunt ejusdem ordinis potentia, et quomodo potentia in actum reducatur, inferius cap. 4 sect. 5 dicetur.

19. Commensurabiles sunt duo simplices ejusdem positionis et ejusdem ordinis actu, quorum uninomia (sepositis signis positionis et ordinis) fuerint commensurabilia.

Ut 3ℭ et 2ℭ sunt commensurabiles, quia 3 et 2 sunt commensurabiles, per cap. 1 sect. 15 Lib. I; item √♀12℞ et √♀3℞ sunt commensurabiles, quia √♀12 et √♀3 sunt commensurabiles, per cap. 1 sect. 14 Lib. I; sic √♀12℞*a*♀ et √♀3℞*a*♀, et similes omnes.

CAPUT II.

DE ADDITIONE ET SUBSTRACTIONE POSITIVORUM.

1. Si positivi addendi vel substrahendi minuendique simplices fuerint atque commensurabiles, tunc uninomia utriusque adde, vel minus à majore substrahe et producto sive residuo postpone signa positiva pristina.

Ut sint addendæ 3℞ ad 2℞, fient 5℞ ; item, 4𝔷 ad 3𝔷 fient 7𝔷 ; item, 6*a*𝔠 ad 9*a*𝔠 fient 15*a*𝔠 ; item, √𝔷2𝔠 ad √𝔷8𝔠 fient √𝔷18𝔠, quia √𝔷2 addita ad √𝔷8 facit √𝔷18, per cap. 2 sect. 1 Lib. I. Sic √𝔷2℞*a*𝔷 ad √𝔷8℞*a*𝔷 facit √𝔷18℞*a*𝔷 ; item, $\frac{\surd\mathfrak{z}2\mathfrak{c}}{3}$ ad $\frac{\surd\mathfrak{z}8\mathfrak{c}}{5}$ facit $\frac{\surd\mathfrak{z}242\mathfrak{c}}{15}$, quia eorum uninomia, per cap. 2 sect. 1 Lib. I, et cap. 13 Lib. I, reducta ad eandem denominationem faciunt $\frac{\surd\mathfrak{z}50}{15}$ et $\frac{\surd\mathfrak{z}72}{15}$, et addita faciunt $\frac{\surd\mathfrak{z}242}{15}$. Item, sint substrahendæ 3℞ ex 5℞, remanent 2℞ ; item, 3*b*𝔷𝔷 ex 8*b*𝔷𝔷, remanent 5*b*𝔷𝔷 ; item, √𝔠3*ß* ex √𝔠192*ß* relinquunt √𝔠81*ß*, per hanc, et cap. 3 sect. 1 Lib. I. Sic √𝔠3𝔷*a*𝔷 ex √𝔠192𝔷*a*𝔷 relinquunt √𝔠81𝔷*a*𝔷.

2. Si simplices incommensurabiles fuerint, interpone copulam + in additione, et copulam — in substractione.

Ut sint addendæ 3℞ ad 2𝔷 fient 2𝔷+3℞ ; item, 4𝔷 ad 2*a*𝔷 fient 4𝔷+2*a*𝔷 ; item, √𝔷5𝔠 ad √𝔷10𝔠 fient √𝔷10𝔠+√𝔷5𝔠 ; item 5*a*𝔷*b* ad 7*a*𝔷*b*𝔷 sunt 7*a*𝔷*b*𝔷+5*a*𝔷*b*. Item, sint substrahenda 3𝔷 ex 2*a*𝔷, remanent 2*a*𝔷—3𝔷 ; item, 2*a* ex 3*a*𝔷, remanent 3*a*𝔷—2*a* ; item, √𝔷3𝔷 ex √𝔠12𝔷, remanent √𝔠12𝔷—√𝔷3𝔷 ; sic √𝔷2*a* ex √𝔷2*ab* fiunt √𝔷2*ab*—√𝔷2*a*.

Corollarium.

3. Hinc patet ex simplicium incommensurabilium additione et substractione compositos oriri.

Quod ex superioribus exemplis constat quorum producta compositi sunt.

4. Adduntur autem, substrahuntur, et abbreviantur compositi eisdem regulis quibus plurinomia cap. 8 et 9, et cap. 7 sect. 7 et 8, Lib. I. Et quæ illic de plurinomiis et uninomiis dicuntur, hîc de compositis et simplicibus subintelligantur.

Ut sint addendá √℥2ℭ*b*℥+3℥—2℞+1 ad 5ℭ+√℥8ℭ*b*℥—4℥+3*a*—6: ea primò (per cap. 8 sect. 1 Lib. I.) copulato, et fient √℥2ℭ*b*℥+3℥—2℞+1+5ℭ+√℥8ℭ*b*℥—4℥+3*a*—6; deinde ea (per cap. 7 sect. 7 et 8 Lib. I.) abbreviata faciunt √℥18ℭ*b*℥—1℥—2℞—5+5ℭ+3*a*.

Exemplum substractionis.

Ex hoc novissimo producto √℥18ℭ*b*℥—1℥—2℞—5+5ℭ+3*a* substrahantur hæc √℥2ℭ*b*℥+3℥—2℞+1: primò (per cap. 9 sect. 1) converte copulas et simplices copulato, fientque √℥18ℭ*b*℥—1℥—2℞—5+5ℭ+3*a*—√℥2ℭ*b*℥—3℥+2℞—1; ultimò, hæc abbreviato, et fient √℥8ℭ*b*℥—4℥—6+5ℭ+3*a*, ut superius.

CAPUT III.

DE RADICUM EX SIMPLICIBUS EXTRACTIONE.

1. Omne signum purum tales et tot habet in se radices insitas, quales et quot sint ejus signi characteres, et præter eas nullas.

Ut 0ʓℭ habet in se radicem quadratam, item cubicam, item denique quadrati cubicam, et nullam præterea aliam.

2. Omne signum mixtum tales et tot habet radices insitas, quales et quot fuerint in singulis diversis suis positionibus characteres communes repetitæ, et nullas præterea alias.

Ut 0ʓℭ*ßa*ʓʓℭ habet insitas radices quadratam, quia tam in primâ quàm secundâ positione ejus reperitur signum ʓ, item cubicam eâdem ratione, item denique quadrati cubicam; et præterea nullam, veluti nec supersolidam, quia *ß* non reperitur inter signa secundæ positionis ejus exempli, nec quadrati quadratam, quia ʓʓ non reperitur inter signa prioris positionis ejusdem exempli.

3. Ex signo puro radicem insitam extrahere, est numerum ordinis signi puri per numerum ordinis qualitatis radicis dividere, et quotientis signum ordinis notare.

Ut sit extrahenda radix cubica ex 0ʓℭ, numerus ordinis ʓℭ est 6, quæ divisa per numerum ordinis cubici, viz., per 3, fit quotiens 2, cujus signum ordinis est ʓ; fit ergo 0ʓ radix cubica hujus 0ʓℭ; sic ejusdem 0ʓℭ radix quadrata est 0ℭ; item ejusdem radix quadrati cubica est 0℞.

4. Ex signo mixto radix aliqua insita extrahitur quum (per præmissam) ex suis singulis diversis positionibus radix talis extrahitur.

Ut sit extrahenda radix cubica ex 0ꝗℭßaꝗꝗℭ : primò, (per præcedentem) ex 0ꝗℭß extrahatur radix cubica, eaque erit 0ꝗß; deinde extrahatur (per eandem) radix cubica ex aꝗꝗℭ, eaque erit aꝗꝗ ; unde et tota radix cubica hujus 0ꝗℭßaꝗꝗℭ erit 0ꝗßaꝗꝗ. Sic ejusdem exempli radix quadrata erit hæc, 0ℭßaꝗℭ ; item ejusdem radix quadrati cubica erit 0ßaꝗ.

5. Si simplicis totius radix quævis extrahenda sit, ejusque simplicis non solum absolutus numerus complectatur talem radicem, sed et signum positivum ejus talem habeat (per sect. 1 et 2, hujus) sibi insitam ; tunc extrahe radicem illam ex numero, eique ascribe radicem illam signi retento priori (si quod fuit) radicali.

Ut sit extrahenda radix cubica ex 64ꝗℭ : primò, radix cubica numeri absoluti erit 4, deinde radix cubica signi ꝗℭ erit ꝗ, per 1 hujus ; tota ergo radix cubica horum 64ꝗℭ erit 4ꝗ : item eorundem 64ꝗℭ radix quadrata erit 8ℭ : item eorundem radix quadrati cubica erit 2℞ : simili modo et radix quadrata hujus √ℭ9ꝗaꝗß est √ℭ3℞aß.

Corollarium.

6. Hinc fit quod simplices habentes et in numero et in signis positivis talem radicem insitam, qualem suum vel totale vel particulare radicale indicat, abbreviantur delendo radicale illud, et extrahendo (per præcedentem) radicem illam ex reliquo.

Ut, sit ille simplex √ꝗℭ4ꝗß, qui sic abbreviatur ; dele particulare radicale ꝗ, remanet √ℭ4ꝗß, cujus (per præmissam) extrahe radicem talem, viz., quadratam, eaque erit √ℭ2ß pro abbreviationis producto, idem valente quod √ꝗℭ4ꝗß. Item,

√𝔮ℭ64𝔮ℭ sic abbreviatur; dele totale radicale √𝔮ℭ, remanet 64𝔮ℭ, cujus (per præmissam) extrahe talem radicem, viz., quadrati cubicam, eaque erit 2℞, quæ idem valet ac √𝔮ℭ64𝔮ℭ.

7. Si simplicis (cujus radix aliqua sit extrahenda) et numero absoluto et signo positivo talis radix non fuerit insita, tunc toti simplici præpone signum radicale radicem illam denotans.

Ut sit extrahenda radix quadrata hujus 4ℭ, ea sit √𝔮4ℭ; item, radix cubica 4ℭ sit √ℭ4ℭ; item, radix cubica hujus √𝔮3℞ erit √𝔮ℭ3℞; item, radix quadrata horum 4𝔮*a* erit √𝔮4𝔮*a*.

CAPUT IV.

DE SIMPLICIUM IN SE MULTIPLICATIONE, ET DE REDUCTIONE.

1. Multiplicare signum purum in se quadrate vel cubice, vel ad alium ordinem, est utriusque ordinis numeros invicem ducere, et producti signum ordinis notare.

Ut sit 0𝔮 (cujus numerus ordinis est 2) multiplicandum in se cubice (cujus cubi numerus ordinis est 3:) duc ergo 2 in 3, producuntur 6, quorum signum ordinis est 𝔮ℭ, est ergo 0𝔮ℭ cubus hujus 0𝔮; item, 0𝔮 supersolide in se ductum facit 0𝔮*ß*; item, 0℞ in se quadrati-cubice ducta facit 0𝔮ℭ.

2. Multiplicare signum mixtum in se ad aliquem ordinem, est signa singularum positionum in se ad illum ordinem (per præmissam) ducere.

Ut sit 0*ßa*𝔮 in se quadrati-cubice ducenda: primò (per præcedentem 1) duc 1*ß* in se quadrati-cubice, fiet 0𝔮ℭ*ß*, deinde

duc *a*℥ in se quadrati-cubice, fiet *a*℥℥𝔠, et per consequens totum 0℥𝔠*βa*℥℥𝔠 erit quadrati cubus hujus 0*βa*℥.

3. Si ergo totum simplicem in se multiplicare volueris quadrate, vel cubice, vel ad alium ordinem, primo tam ejus numerum absolutum arithmetice, quàm ejus signum (per 1 et 2 hujus) in se ad illum ordinem duc, deinde producti radicem talem extrahe (per cap. 3 sect. 5 et 7) qualem suum radicale (si quod sit) denotat.

Ut sint 3𝔠 in se quadrate multiplicandi: duc ergo 3 in se quadrate, per Arithmeticam, et 𝔠 in se quadrate (per 1 hujus), fient 9℥𝔠 pro vero quadrato trium cuborum. Item, 2℥℥ in se cubice ducta faciunt 8℥℥𝔠. Item, √𝔠3℞ in se quadrate ducta facit √𝔠9℥. Item, sit √℥2℞*a*℥*c* in se quadrate ducenda; primò quadrentur 2 et ℞*a*℥*c*, fientque (per 2 hujus) 4℥*a*℥℥*c*℥, cujus extrahatur radix quam radicale indicat, viz., quadrata, ea erit (per cap. 3 sect. 5) 2℞*a*℥*c* pro vero quadrato hujus √℥2℞*a*℥*c*, quæ quidem √℥2℞*a*℥*c* etiam facilius (per cap. 6 sect. 6 Lib. I.) deleto radicali quadratur.

4. Si simplices dissimiliter radicati, ad similia radicalia sint reducendi, uniuscujusque partem rationalem toties (per præcedentem) in se duc quoties cæterorum dissimilia omnia radicalia indicant, et unicuique producto per se posito, cuncta dissimilia radicalia, una cum communi et simili (si quod sit) radicali, præpone.

Ut √𝔠3℞ et √℥2℞ sic reducuntur: duc 3℞ in se quadrate, et 2℞ in se cubice, fientque 9℥ et 8𝔠; quibus utrumque radicale dissimile præpone, fientque √℥𝔠9℥ et √℥𝔠8𝔠, ejusdem nempe radicalis, valoris autem ejusdem cujus √𝔠3℞ et √℥2℞: Item, √℥6℥ et 2℞ sic reducuntur; duc 2℞ in se quadrate, quia in priore est hoc radicale √℥, at 6℥ non multiplicantur, quia 2℞

carent radicali; fiunt ergo 6♀ et 4♀, quibus illud unicum radicale præpone, fientque √♀6♀ et √♀4♀, ejusdem radicalis et præterea ejusdem ordinis actu: Item, √♀♀2♀*b* et √♀ᘓ3*ac* sic reducuntur; duc 2♀*b* in se cubice, et 3*ac* in se quadrate, (quia eorum radicalia differunt in cubo et altero quadratorum, et in altero quadratorum conveniunt), fiunt 8♀ᘓ*b*ᘓ et 9*a*♀*c*♀, quibus præpone illud commune radicale √♀, una cum dissimilibus ♀ et ᘓ, fient √♀♀ᘓ8♀ᘓ*b*ᘓ et √♀♀ᘓ9*a*♀*c*♀, ejusdem radicalis, et valoris pristini.

Exemplum plurium duobus reducendorum.

Sint hæc tria √♀2♀, √ᘓ3℞, et √♀♀1*a* reducenda ad idem radicale: duc primò 2♀ in se quadrati-cubice, non autem quadrati quadrati cubice (quia in altero quadratorum convenit primum cum ultimo, non autem in reliquo quadrato), fiet ergo primum, cum suis radicalibus dictis, √♀♀ᘓ64♀♀ᘓ; secundum autem simili ratione ductum quadrati-quadrate, faciet suum productum, cum radicalibus debitis, √♀♀ᘓ81♀♀; tertium denique ductum cubice tantum et adhibitis suis radicalibus fiet √♀♀ᘓ1*a*ᘓ; quæ quidem tria sunt ejusdem jam radicalis; et sic de reliquis.

Corollarium.

5. Hinc sequitur quod quidam simplices dissimiliter radicati ejusdem ordinis potentia dicti, per reductionem fiunt ejusdem ordinis actu; quidam autem non, videlicet quorum alterum majoris, alterum minoris ordinis fiunt.

Ut in superioribus exemplis √♀6♀ et 2℞ sunt ejusdem ordinis potentia, quia reducti constituunt √♀6♀ et √♀4♀, qui (per

cap. 1 sect. 18 Lib. II.) sunt ejusdem ordinis actu; at √Ꞓ3℞ et √ℨ2℞ reducti constituunt √ℨꞒ9ℨ et √ℨꞒ8Ꞓ, quorum alterum, viz., √ℨꞒ8Ꞓ est altioris seu majoris ordinis, alterum vero √ℨꞒ9ℨ est minoris.

CAPUT V.

DE POSITIVORUM MULTIPLICATIONE GENERALI.

1. Multiplicare signa ejusdem positionis invicem est numeros ordinum eorum signorum addere, et signum ordinis producti numeri notare.

Ut sit multiplicandum 0*a*ℨ per 0*a*Ꞓ, numeri ordinum *a*ℨ et *a*Ꞓ sunt 2 et 3, quæ addita faciunt 5, quorum signum ordinis est *aß*; est ergo 0*aß* productum multiplicationis 0*a*ℨ in 0*a*Ꞓ. Sic 0*b* ductum per 0*b*Ꞓ, facit 0*b*ℨℨ. Item 0ℨ per 0ß fiet 0ſß.

2. Multiplicare signa pura diversarum positionum invicem, est ipsa signa simul connexa scribere, præposito semper signo primæ positionis (si quod sit) ante reliqua.

Ut sit multiplicandum 0*a*ℨ per 0*b*Ꞓ, producetur 0*a*ℨ*b*Ꞓ, cujus pronuntiatio, quum numerum habet ut 6*a*ℨ*b*Ꞓ, est hæc, 6*a*ℨ ducta per 1*b*Ꞓ. Item, 0*a*ℨ per 0Ꞓ non producit 0*a*ℨꞒ sed 0Ꞓ*a*ℨ, præposito signo primæ positionis; quod quidem 0Ꞓ*a*ℨ sic pronuntiatur, tot seu nulli cubi primæ positionis ducti in unum quadratum secundæ. At contra, 0*a*ℨꞒ ex toto est secundæ tantum positionis (ut ex Tabulâ cap. 1 hujus patet), et sic pronuntiatur, unum *a* quadrati cubicum.

3. Hinc sequitur, ex signis puris diversarum positionum invicem ductis produci mixta.

Ut in exemplo superiore, purum 0*a*ꝗ per purum 0*b*ꞇ ductum producit 0*a*ꝗ*b*ꞇ, per præcedentem; quod (per cap. 1 sect. 14) est mixtum.

4. Mixta etiam, quatenus communicant invicem positiones, per sectionem primam hujus multiplicantur; quatenus autem sunt diversarum positionum, multiplicantur per secundam hujus.

Ut 0℞*a*ꝗ per 0ꞇ*a* ductum facit 0ꝗꝗ*a*ꞇ; ductis nempe (per 1 hujus) similibus positionibus invicem, viz. 0℞ per 0ꞇ, et *a*ꝗ per *a*, fient 0ꝗꝗ et *a*ꞇ; et rursus ꝗꝗ ductum per *a*ꞇ fit (per 2 hujus) 0ꝗꝗ*a*ꞇ. Item, 0℞*a*ꝗ per 0ꞇ*b* ductum facit 0ꝗꝗ*a*ꝗ*b*; quia 0℞ per 0ꞇ facit 0ꝗꝗ, et 0ꝗꝗ per *a*ꝗ facit (per 2 hujus) 0ꝗꝗ*a*ꝗ, et 0ꝗꝗ*a*ꝗ per 0*b* facit (per eandem 2) 0ꝗꝗ*a*ꝗ*b*.

5. Si simplices invicem ducendi sint: primò, fiant consimiliter radicati, saltem per reductionem (cap. 4 sect. 4); deinde, duc utriusque numeros absolutos invicem; tertiò, duc ad invicem signa positiva per jam dicta; quartò, numeri signique producti radicem talem extrahe qualem indicat suum radicale, (per cap. 3) et huic tandem præpone debitam copulam, (per cap. 6 Lib. I.)

Ut 2*a*ꝗ per 5*a*ꞇ ducta faciunt 10*a*ß, quia 2 per 5 ducta faciunt 10, et *a*ꝗ per *a*ꞇ facit *a*ß. Sic 2*a*ꝗ per 5ꞇ ducta faciunt 10ꞇ*a*ꝗ. Item, √ꝗ2℞*a*ꝗ per √ꝗ8℞ ducta facit 4℞*a* productum; quia ductis numeris invicem, et signis invicem, fiunt 16ꝗ*a*ꝗ, quorum radix quam indicat radicale, viz. radix quadrata, est 4℞*a*. Item, sint ducenda 2℞*a* per —√ꝗ3*ab*; primò per reductionem fiant √ꝗ4ꝗ*a*ꝗ, et —√ꝗ3*ab*; deinde ductis numeris invicem, et signis positivis invicem, fiunt 12ꝗ*a*ꞇ*b*,

quorum radix quadrata cum debitâ copulâ est —√℥12℥aℭb, pro multiplicationis producto.

6. Si compositi ad invicem ducendi sint; singulos multiplicandi simplices duc in singulos multiplicantis; productum autem per commensurabilium additionem et substractionem abbrevia, modo, quo plurinomia ducuntur et abbreviantur, per cap. 7, sect. 7 et 8; et cap. 10, sect. 1 Lib. I.

Ut sint multiplicanda 2a℥+√℥3℞—2℞a—4 per √℥12ℭ +2℥, duc simplices singulos in singulos, fientque √℥48ℭa℥℥ +6℥ — √℥48ßa℥ —√℥192ℭ+4℥a℥ + √℥12ß — 4ℭa—8℥, quæ inde per abbreviationem fiunt √℥48ℭa℥—2℥—√℥48ßa℥ —√℥192ℭ+4℥a℥+√℥12ß—4ℭa.

CAPUT VI.

DE SITU ET COLLOCATIONE SIMPLICIUM COMPOSITI.

1. Intervallum ordinum est differentia inter ordines simplicium ejusdem radicalis, quâ numerus majoris ordinis exsuperat numerum minoris ordinis proxime sequentis.

Ut in hoc composito √ℭ3℞—√ℭ2℥, intervallum est 1; quia numerus ordinis ℞ est 1, et numerus ordinis ℥ est 2, quorum differentia est 1. Item in hoc similiter radicato √℥3ℭ+√℥2℞ —√℥5, intervallum inter √℥3ℭ et √℥2℞ est 2, quia differentia inter numeros ordinum eorum est 2. Sic intervallum inter √℥2℞ et √℥5 est 1, quia enim numerus ordinis ℞ est 1, et numerus ordinis numeri simplicis est 0; eorum ergo differentia seu intervallum erit 1.

2. Simplex fictus alicujus ordinis est nihil, seu 0, ornatum signis radicalibus et positivis illius ordinis.

Ut simplex fictus ordinis cubici est 0ℭ. Item simplex fictus ordinis quadrati est 0𝔮. Item simplex fictus pro radice quadratâ aliquorum cuborum est √𝔮0ℭ. Item simplex fictus ponendus pro radice cubicâ supersolidorum est √ℭ0ß.

3. Intervalla redduntur eadem, cum (per 1, 2, et 3 Lib. VII. Euclidis,) maxima communis mensura dividens illa capitur, atque intervallo hujus mensuræ à maximo ordine ad minimum progrediuntur simplices veri, aut (ubi veri desunt) ficti.

Ut in hoc superiore exemplo, √𝔮3ℭ+√𝔮2℞—√𝔮5 sunt (per 1 hujus,) duo diversa intervalla, viz. 2 et 1, quorum 2 et 1 maxima communis mensura, ex Euclide Lib. VII. prop. 1, est unitas: hâc ergo unitate tanquam intervallo à maximo ordine, viz. à √𝔮3ℭ progredere ad minimum, viz. √𝔮5: hoc modo, substrahe à numero ordinis hujus √𝔮3ℭ unitatem, fit √𝔮0𝔮 fictus simplex, quia verus deest. Item ex numero ordinis hujus √𝔮0𝔮 aufer 1, fit ordo hujus √𝔮2℞; ex cujus denique numero ordinis aufer 1, fit ordo minimus, viz. hujus √𝔮5. Sic ergo collocabis simplices præfati exempli √𝔮3ℭ+√𝔮0𝔮+√𝔮2℞—√𝔮5, et omnia intervalla erunt eadem. Item in hoc 1ℭ—3℞—6 maxima etiam mensura communis est 1, per quam unitatem fit hæc progressio 1ℭ+0𝔮—3℞—6, et erunt intervalla eadem. Item in hoc √ℭ1𝔮ℭ—√ℭ3℞+√ℭ8 erunt intervalla diversa, viz. 5 et 1, quorum unitas adhuc erit communis mensura. Sic ergo (factâ per substractionem unitatis progressione,) collocabis ordines √ℭ1𝔮ℭ+√ℭ0ß+√ℭ0𝔮𝔮+√ℭ0ℭ+√ℭ0𝔮—√ℭ3℞+√ℭ8. Item in hoc √ℭ2ʃʃß+√ℭ3ß—√ℭ10℞ erunt intervalla 6 et 4, et communis mensura 6 et 4 est binarius; facto

igitur progressu per binarium, hic erit situs, √ℭ2ſſß+√ℭ0ℭℭ +√ℭ0ſß+√ℭ3ß+√ℭ0ℭ—√ℭ10℞, quorum intervalla eadem sunt, viz. binario constant.

4. Ut ergo simplices compositorum recte collocentur; primò, fiant omnes consimiliter radicati, per cap. 4, sect. 4; deinde, quæ ejusdem sunt ordinis simul ponuntur, et qui majores sunt ordines minoribus anteponuntur; tertiò, omnia intervalla fiant (per præcedentem) eadem: ultimò, simplices abbreviabiles (si libet,) abbreviare poteris per hujus cap. 3, sect. 6.

Ut 2—√ℭ3℞+1♀+√ℭ2♀ℭ facta ejusdem radicalis, fient √ℭ8—√ℭ3℞+√ℭ1♀ℭ+√ℭ2♀ℭ; deinde, præpositis majoribus ordinibus, et simul positis eis qui ejusdem sunt ordinis, fient √ℭ2♀ℭ+√ℭ1♀ℭ—√ℭ3℞+√ℭ8; tertiò, factis intervallis eisdem fient √ℭ2♀ℭ+√ℭ1♀ℭ+√ℭ0ß+√ℭ0♀♀+√ℭ0ℭ +√ℭ0♀—√ℭ3℞+√ℭ8; ultimò, hæc (si libet) abbreviabis, fientque √ℭ2♀ℭ+1♀+√ℭ0ß+√ℭ0♀♀+0℞+√ℭ0♀—√ℭ 3℞+2. Item eisdem rationibus 1♀+√♀2℞—3 recte sic collocantur, 1♀+√♀0ℭ+0℞+√♀2℞—3: et sic de cæteris.

5. Hinc fit quod compositorum simplices recte, per præcedentem, collocati, ordinem servant proportionalem; singuli scilicet intermedii simplicis quadratum ejusdem erit ordinis (potentiâ saltem,) cujus fuerit productum quod fit ex proxime præcedente in proxime subsequentem ducto.

Ut in hoc ultimo exemplo, præcedente 1♀ [i] +√♀0ℭ [ii] +0℞ [iii] + √♀2℞ [iiii] —3 [v], quadratum secundi est 0ℭ, et duc 1♀ in 0℞, fit etiam 0ℭ. Item quadratum tertii erit 0♀, et duc √♀0ℭ in √♀2℞, fit etiam 0♀, seu (quod idem est) √♀0♀♀. Item quad-

ratum quarti erit 2℞, et duc 0℞ in 3, fit 0℞; atque 2℞ et 0℞ sunt ejusdem ordinis: unde in similibus ordines semper proportionales dicuntur.

6. Hinc etiam sequitur (quum minores ordines posterius collocentur) numerum simplicem et absolutum (quia est nullius positivi ordinis) ultimo omnium loco collocandum.

7. In compositis mixtis vel plurium positionum, quot fuerint positiones diversæ, tot aggregati positionum characteres ponuntur pro rebus, seu rerum ordine, ac tot etiam rerum characteres in se, et invicem ducti, et aggregati, faciunt ordinem integrum quadratorum, ac singuli illi characteres rerum in singulos hos quadratorum ducti, et aggregati, faciunt integrum ordinem cuborum, et sic deinceps in infinitum.

Ut in 1♀—1℞+1*a*♀+1*a*—18, duæ sunt positiones diversæ, viz. R et *a*; igitur —1℞+1*a* pro ordine uno, viz. rerum ponitur. Item duc (per cap. 5,) —℞+*a* in se, fient +♀—0℞*a*+♀, atque igitur 1♀—0℞*a*+1*a*♀, licet sint tres simplices, efficiunt unicum tantum ordinem, quadratorum nempe. Est ergo illius compositi hic situs rectus 1♀—0℞*a*+1*a*♀—1℞+1*a*—18. Simili ratione ordo cubicus hujus exempli esset hic.

CAPUT VII.

DE DIVISIONE.

1. Dividere signum purum majus per minus signum, ejusdem positionis, est signum ordinis intervalli eorundem post numerum simplicem, aut post ejus numeratorem, collocare.

Ut sit dividendum 0*a*ℭ per 0*a*℥ ; signum intervalli est *a*℞, quod post locum saltem numeri simplicis colloca, vel post ejus numeratorem, ut libet, fitque 0*a*℞, vel $\frac{0a℞}{0}$, pro quotiente divisionis. Item 0ß per 0ℭ fit quotiens 0℥, vel $\frac{0℥}{0}$.

2. Dividere signum purum minus per majus ejusdem positionis, est signum intervalli eorundem post numeri simplicis fracti denominatorem collocare, et quotiens semper erit fractio.

Ut sit dividendum 0*a*℥ per 0*a*ℭ ; signum intervalli est *a*, quod post locum saltem denominatoris numeri simplicis colloca, fitque $\frac{0}{0a℞}$ pro quotiente fracto. Item 0ℭ per 0ß divisum facit quotientem $\frac{0}{0℥}$.

3. Dividere signum purum per alterius positionis signum, est signum dividendum superscribere, (viz. post numeratorem numeri simplicis fracti,) et signum dividens subscribere, (viz. post ejusdem denominatorem,) et quotiens semper erit fractio.

Ut sit 0*a*ℭ per 0℥ dividendum, fit quotiens $\frac{0aℭ}{0℥}$. Item sit 0℥ dividendum per 0*a*ℭ, fit quotiens $\frac{0℥}{0aℭ}$.

Corollarium.

4. Hinc sequitur, in mixtis divisionem debere fieri per sect. 1 et 2,

quatenus communicant positiones; quatenus autem sunt diversarum positionum, per sect. 3.

Ut sit 0ℭ*a*𝔔 dividendum per 0𝔔*a*; divide (per sect. 1) 0ℭ per 0𝔔, fit 0℞. Item *a*𝔔 per *a*, fit *a*; fit ergo totus quotiens 0℞*a*. Item 0ℭ*a*𝔔 per 0ß*b* dividitur hoc modo, 0𝔔 per 0ß dividatur (per sect. 2,) et fit $\frac{0}{0𝔔}$. Item, *a*𝔔 dividatur per *b* (per sect. 3) et fit $\frac{𝔔}{b}$; fit ergo pro toto quotiente $\frac{0a𝔔}{0𝔔b}$. Item sit 0*b*𝔔*c* per 0*bc*𝔔 dividendum, fit quotiens $\frac{0b}{0c}$.

5. Ex omni integro fit fractio ejusdem valoris subscribendo unitatem, et interponendo lineam.

Ut 5 sunt numerus integer, et ex eo fit $\frac{5}{1}$ fractio. Item √ℭ7 est integra, et ex eâ fit $\frac{√ℭ7}{1}$ fractio.

6. Si ergo simplex per simplicem dividendus fuerit; primò, fiant consimiliter radicati, (per cap. 4, sect. 4;) deinde, divide numerum simplicem dividendi per numerum simplicem divisoris; tertiò, (per jam dicta,) divide signum positivum dividendi per signum divisoris; quartò, numeri signique pro quotiente producti radicem talem extrahe, qualem indicat suum radicale, (per cap. 3,) et huic tandem præpone debitam copulam, per cap. 6 Lib. I.

Ut 12*b*ℭ sint dividenda per 3*b*𝔔; divide 12 per 3, fiunt 4; et divide *b*ℭ per *b*𝔔, fit *b*; erit ergo totus quotiens 4*b*. Item sint √𝔔20ß dividenda per √𝔔8ℭ; divide numerum per numerum, fit $\frac{5}{2}$, et divide signum per signum, et fit 𝔔. Ex his ergo $\frac{5}{2}$𝔔, seu $\frac{5𝔔}{2}$, seu $2\frac{1}{2}$𝔔, (quæ per 1 hujus eadem sunt,) extrahe radicem quadratam; ea erit $\frac{√𝔔5𝔔}{√𝔔2}$, seu √𝔔$\frac{5}{2}$𝔔, seu √𝔔$2\frac{1}{2}$𝔔 pro quotiente; suntque hæc eadem, per cap. 4, sect. 4 Lib. I. Item, è converso, sit dividenda √𝔔8ℭ per √𝔔20ß, erit quotiens $\frac{√𝔔2}{√𝔔5𝔔}$, seu rectius √𝔔$\frac{2}{5}$𝔔. Item, sit dividenda —√𝔔12𝔔*a*ℭ*b* per

$\sqrt{}$♀$3ab$: divide numeros et signa ut dictum est, fiet 4♀a♀, quorum radix quam indicat radicale, viz. quadrata, est —2℞a pro quotiente, cum suâ debitâ copulâ. Item, sint 4℞a dividenda per —$\sqrt{}$♀2℞a♀ : primò, fiant ejusdem radicalis, sic, $\sqrt{}$♀16♀a♀ et —$\sqrt{}$♀2℞a♀ ; deinde, illius et numerum et signum per hujus divide, et fit 8℞, quorum radix quadrata, cum debitâ copulâ, est —$\sqrt{}$♀8℞ pro quotiente.

7. Hîc observandum, quod si quotiens signorum positivorum fuerit (per 2 et 3 hujus) fractio, et quotiens numeri absoluti fuerit integer, ex hoc integro fieri debet (per 5 hujus) fractio.

Ut si sint 12♉ dividenda per 3ß ; divide numerum per numerum, fiunt 4, numerus integer ; et divide signum per signum, fit fractio, nempe $\frac{0}{0\text{♀}}$, seu $\frac{}{\text{♀}}$; atque igitur ex integris 4, seu $\frac{4}{1}$, et fiet totus quotiens $\frac{4}{1\text{♀}}$. Item sint $15b$♀c per $5bc$♀ dividenda, fit quotiens numeri divisionis 3 integer, et quotiens signorum $\frac{0b}{0c}$, seu $\frac{b}{c}$ quæ fractio est, atque igitur ex 3 fiat fractio $\frac{3}{1}$ et fiet totus quotiens $\frac{3b}{1c}$, et non $3\frac{b}{c}$ nec $\frac{3b}{0\text{♀}}$.

8. Si compositus per simplicem dividendus fuerit, divide (per jam dicta) quamvis particulam simplicem compositi per hunc simplicem divisorem, et quotientis simplices debitis copulis connecte.

Ut sint 12♀—$\sqrt{}$♀2♀+6 dividenda per 2℞ ; divide 12♀ per 2℞, fiunt 6℞ ; item divide —$\sqrt{}$♀2♀ per 2℞, fiunt —$\sqrt{}$♀$\frac{1}{2}$; denique divide +6 per 2℞, fiunt $\frac{+3}{1\text{℞}}$; quibus copulatis fit totus quotiens 6℞—$\sqrt{}$♀$\frac{1}{2}+\frac{3}{1\text{℞}}$.

(Si per compositum unius ordinis sit dividendum quippiam &c., ut sit dividendum per 2℞—$\sqrt{}$♀3♀, fac ut cap. 11, sect. 2 Lib. I.)

9. Si compositus per compositum plurium ordinum dividendus fuerit: primò, utriusque compositi simplices situ recto (per cap. 6 sect. 4) collocentur; deinde simplicem maximi ordinis dividendi per simplicem maximi ordinis divisoris divide, (per 6 hujus,) et producetur simplex primus quotientis; per hunc multiplica totum divisorem, productum ex toto dividendo aufer, reliquias nota, deleto cætero dividendo. Ex his reliquiis conficito alium dividendum, ex quo, eodem quo prius modo, alium quotientis simplicem, aliasque forsan reliquias, producito, donec tandem aut nihil relinquitur dividendi, aut reliquiæ saltem paucioribus constent ordinibus proportionalibus quàm divisor; quibus peractis, collige et connecte singulas dictas particulas quotientis cum copulis suis, et fiet quotiens totalis, observatis etiam reliquiis ultimis si quæ sint.

Ut sint 1ℨℨ+71ℨ+120—154℞—14ℭ dividenda per 6+1ℨ—5℞: primò, (per cap. 6,) recte collocentur hoc situ:—

1ℨℨ—14ℭ+71ℨ—154℞+120 (1ℨ
1ℨ— 5℞+6

Deinde divide 1ℨℨ per 1ℨ, fit quotiens 1ℨ, quod apud hemicyclum nota, ut supra; per hunc quotientem duc totum divisorem, fit inde 1ℨℨ—5ℭ+6ℨ, quæ ex toto dividendo substrahe; relinquentur —9ℭ+65ℨ—154℞+120; hæc igitur nota, deletis cæteris hâc formâ:—

— 9ℭ+65ℨ
1ℨℨ—14ℭ+71ℨ—154℞+120 (1ℨ
1ℨ — 5℞+ 6
1ℨ — 5℞+6

Has reliquias per eundem divisorem modo quo prius divide, et fiet hic situs :—

```
            +20ℨ
       — 9ℭ+65ℨ—100℞
  1ℨℨ—14ℭ+71ℨ—154℞+120 ( 1ℨ—5℞
  1ℨ  — 5℞+ 6  +  6  +6
        1ℨ— 5℞+ 5℞
            1ℨ
```

Has denique reliquias eodem quo prius modo divide, et fiet hic situs :—

```
          +20ℨ
      — 9ℭ+65ℨ—100℞
  1ℨℨ—14ℭ+71ℨ—154℞+120 ( 1ℨ—9℞+20
   1ℨ— 5℞+ 6 + 6+6
       1ℨ— 5℞—5℞
           1ℨ
```

Quotiens ergo totalis est 1ℨ—9℞+20, et novissimæ reliquiæ nullæ supersunt.

Aliud exemplum unico typo expressum, in quo 1ℭ—11ℨ+40℞—36 dividitur per 1℞—4.

```
            + 12
        +12℞
    — 7ℨ
  1ℭ—11ℨ+40℞—36 ( 1ℨ—7℞+12 pro quotiente totali.
  1℞— 4
      1℞ — 4
          1℞ — 4
```

Et jam ultimo supersunt 12, pauciores habentes ordines quàm divisor 1℞—4.

10. Si dividendus non fuerit integre seu totaliter (viz. sine residuo ultimo,) divisibilis per divisorem, tunc subscribendo divisorem sub residuo fiat fractio, quam apud quotientem colloca, aut si mavis sub toto dividendo subscribe divisorem.

Ut in superiore exemplo, erit quotiens verus 1♀—7℞ +12$\frac{+12}{1\text{℞}-4}$, aut si mavis $\frac{1\text{𝔠}-11\text{♀}+40\text{℞}-36}{1\text{℞}-4}$; fitque hoc quia divisor ille nequit dividendum integre, et sine residuo dividere.

CAPUT VIII.

DE RADICUM EX COMPOSITIS EXTRACTIONE.

1. Si compositi radix quadrata fuerit extrahenda, compositus ille primò recte (per cap. 6 sect. 4) collocetur; deinde, ex maximi ordinis simplice vel simplicibus (per cap. 3 sect. 5 et 7,) radicem quadratam extrahe, quam apud hemicyclum pro quotiente constitue, et illa simplicia maximi ordinis dele, nullo, si possis, relicto residuo. Secundò, per totius quotientis duplum divide primam partem compositi remanentem (per cap. 7 sect. 9), nullo, si poteris, relicto residuo. Hujus divisionis quotientem novum post præfatum quotientem scribe, ejusque novi quadratum ex dictâ parte superstite aufer, notatis reliquiis. Hocque secundum opus tertiò, quartò, sæpiusque repete, donec tandem aut nullæ supersint reliquiæ, et tunc totus quotiens cum copulis suis erit vera radix insita quæsita, aut si aliquæ quàm paucissimæ supersint reliquiæ, tunc dictus quotiens radix proxima dicitur, et non vera.

Ut sit ex composito hoc √♀4𝔠+1♀—√♀576℞+144—23℞ extrahenda radix quadrata: primò, recte locentur simplices, hoc situ:—

1♀+√♀4𝔠—♀+√♀4𝔠—23℞—√♀576℞+144;

deinde ex primo, viz. ex 1𝔮 extrahe radicem veram, ea erit 1℞ pro quotiente, et relinquentur cætera, hoc situ :—

1𝔮+√𝔮4𝔠—23℞—√𝔮576℞+144 (1℞ ;

secundò, per quotientis duplum, viz. per 2℞ seu √𝔮4𝔮 divide primam partem reliqui, viz. √𝔮4𝔠, fiet novus quotiens +√𝔮1℞ et reliquiæ —23℞—√𝔮576℞+144, ex quibus aufer hujus novi quotientis quadratum, quod est 1℞, restabunt reliquiæ et quotiens hoc situ :—

—24℞
1𝔮+√𝔮4𝔠—23℞—√𝔮576℞+144 (1℞+√𝔮1℞
+√𝔮4𝔮

Adhuc repete hoc secundum opus, nempe per quotientis duplum, viz. per 2℞+√𝔮4℞ divide partem primam reliquiarum superiorum, viz. —24℞—√𝔮576℞, fiet novissimus quotiens —12, et reliquiæ +144, ex quibus aufer novissimi quotientis quadratum, viz. —144, et nihil relinquitur, ut hoc situ patet :—

—24℞
1𝔮+√𝔮4𝔠—23℞—√𝔮576℞+144 (1℞+√𝔮1℞—12
+√𝔮4𝔮+ 2℞ +√𝔮4℞

Unde patet quod 1℞+√𝔮1℞—12 sunt vera radix quadrata insita hujus compositi suprascripti, quia nihil post extractionem restat.

Aliud exemplum.

Ex √𝔠4𝔮—8—√𝔠16℞ sit extrahenda radix quadrata : primò, fiat hic situs :—

√𝔠4𝔮—√𝔠16℞—8 (√𝔠2℞

Secundò, fiat hic situs :—

$$\begin{array}{l} \qquad\qquad\qquad\qquad -9 \\ \underline{\surd\mathfrak{C}4\text{ʓ}}-\underline{\surd\mathfrak{C}16\text{℞}}-\underline{8}\ (\ \surd\mathfrak{C}2\text{℞}-1 \\ \qquad\quad +\underline{\surd\mathfrak{C}16\text{℞}} \end{array}$$

Unde patet quod $\surd\mathfrak{C}2\text{℞}-1$ est proxima et non vera radix quadrata hujus compositi $\surd\mathfrak{C}4\text{ʓ}-\surd\mathfrak{C}16\text{℞}-8$, quia restant reliquiæ, viz. -9.

Exemplum plurium positionum.

Ex $1\text{ʓ}+2\text{℞}a+1a\text{ʓ}+1\text{℞}+1a-110$ sit extrahenda radix quadrata. Primò, fiat hic situs, (per cap. 6 prop. 7) :—

$$\underline{1\text{ʓ}}+2\text{℞}a+1a\text{ʓ}+1\text{℞}+1a-110\ (\ 1\text{℞}$$

Secundò, fiat hic situs :—

$$\begin{array}{l} \underline{1\text{ʓ}}+\underline{2\text{℞}a}+\underline{1a\text{ʓ}}+1\text{℞}+1a-110\ (\ 1\text{℞}+1a \\ \quad +\underline{2\text{℞}} \end{array}$$

Tertiò, fiat hic situs :—

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad -110\tfrac{1}{4} \\ \underline{1\text{ʓ}}+\underline{2\text{℞}a}+\underline{1a\text{ʓ}}+\underline{1\text{℞}}+\underline{1a}-\underline{110}\quad (\ 1\text{℞}+1a+\tfrac{1}{2}\ \text{pro radice} \\ \quad \underline{2\text{℞}}\qquad\qquad\ \underline{2\text{℞}}+\underline{2a}\qquad\qquad\qquad \text{proximâ.} \end{array}$$

Aliud exemplum plurium positionum.

Sit ex $1\text{ʓ}+6\text{℞}a-7$ extrahenda radix quadrata. Primò, fiat hic situs (per cap. 6 prop. 7) :—

$$\begin{array}{l} \qquad\qquad\quad -9a\text{ʓ}-7 \\ \underline{1\text{ʓ}}+6\text{℞}a+0a\text{ʓ}-7\ (\ 1\text{℞} \end{array}$$

Secundò, fiat hic situs :—

$$\begin{array}{l} \qquad\quad -9a\text{ʓ} \\ \underline{1\text{ʓ}}+\underline{6\text{℞}a}+\underline{0a\text{ʓ}}-7\ (\ 1\text{℞}+3a\ \text{pro radice proximâ.} \\ \quad \underline{2\text{℞}} \end{array}$$

Exemplum difficile unius positionis.

Sit ex $1\text{q}-\sqrt{\text{q}}8\text{q}-6℞+8+\sqrt{\text{q}}32$ extrahenda radix quadrata. Primò, fiat hic situs :—

$$1\text{q}-\sqrt{\text{q}}8\text{q}-6℞+8+\sqrt{\text{q}}32\ (\ 1℞$$

Secundò, fiat hic situs :—

$$\begin{array}{l} \qquad\qquad -3-\sqrt{\text{q}}8 \\ 1\text{q}-\sqrt{\text{q}}8\text{q}-6℞+8+\sqrt{\text{q}}32\ (\ 1℞-\sqrt{\text{q}}2-3 \text{ pro} \\ \quad +\sqrt{\text{q}}4\text{q}+2℞ \qquad\qquad\qquad \text{radice proximâ.} \end{array}$$

Sextum exemplum.

Ex $1\text{q}-0℞a+1a\text{q}-1℞+1a-18$ sit extrahenda radix quadrata. Primò, fiat hic situs, per cap. 6 prop. 7 :—

$$1\text{q}-0℞a+1a\text{q}-1℞+1a-18\ (\ 1℞$$

$$\begin{array}{l} \qquad +2℞a \\ \text{Secundò :—}1\text{q}-0℞a+1a\text{q}-1℞+1a-18\ (\ 1℞-1a \\ \qquad +2℞ \end{array}$$

Tertiò, fiat hic situs :—

$$\begin{array}{l} +2℞a \qquad\qquad\qquad -18\tfrac{1}{4} \\ 1\text{q}-0℞a+1a\text{q}-1℞+1a-18\ \ (\ 1℞-1a-\tfrac{1}{2} \\ +2℞ \qquad\qquad +2℞-2a \end{array}$$

2. Si compositi radix cubica fuerit extrahenda, compositus ille primò recte (per cap. 6 sect. 4) collocetur ; deinde ex maximi ordinis simplice (per cap. 3 sect. 5 et 7) radicem cubicam extrahe, quam apud hemicyclum pro quotiente constitue, et illum simplicem maximi ordinis dele. Secundò, per tria quadrata totius quotientis divide primam partem non deletam compositi (per cap. 7 sect. 9), deletis divisis, nota reliquias. Hujus divisionis quotientem novum post primum quotientem scribe,

ejusque novi tria quadrata ducta in primum antecedentem quotientem ex dictis reliquiis aufer, et ex eisdem aufer cubum novi, reliquiis notatis. Hocque secundum opus iterum atque iterum repete, donec tandem aut nullæ supersint reliquiæ, et tunc totus quotiens cum copulis suis erit vera radix insita quæsita; aut si aliquæ quàm paucissimæ supersint reliquiæ, tunc dictus quotiens dicitur radix proxima, et non vera.

Ut, sit sequentis compositi extrahenda radix cubica, qui sic primò recte collocetur :—

1Qℭ+12ß+60QQ+160ℭ+240Q+192℞+64 (1Q

viz. extrahitur ex 1Qℭ radix cubica, quæ est 1Q, quod pro quotiente ponitur. Secundò, per tria quadrata quotientis, viz. per 3QQ, divide 12ß, fiet novus quotiens +4℞ deletis 12ß, hoc situ :—

1Qℭ+12ß+60QQ+160ℭ+240Q+192℞+64 (1Q+4℞
+3QQ

Deinde, hujus novi quotientis +4℞ tria quadrata, viz. 48Q, duc in priorem quotientem, viz. in 1Q, fient 48QQ, quæ aufer ex 60QQ &c., restant +12QQ+160ℭ+240Q+192℞+64, ex quibus etiam reliquiis aufer cubum horum +4℞, qui est 64ℭ, restant +12QQ+96ℭ+240Q+192℞+64, hoc situ :—

+12QQ+96ℭ
1Qℭ+12ß+60QQ+160ℭ+240Q+192℞+64 (1Q+4℞
3QQ+48QQ+64ℭ

Tertiò, repete secundum opus, viz. per tria quadrata quotientis, quæ sunt 3QQ+24ℭ+48Q, divide primam partem dictarum reliquiarum, fient quotiens et reliquiæ ut infra :—

+12QQ+ 96ℭ+ 48Q
1Qℭ+12ß +60QQ+160ℭ+240Q+192℞+64 (1Q+4℞+4
3QQ+48QQ+ 64ℭ
+ 3QQ+ 24ℭ+48Q

Deinde, hujus novissimi quotientis tria quadrata, viz. 48, duc in totum antecedentem quotientem, viz. in 1ℨ+4℞, fient 48ℨ +192℞, quæ ex reliquiis illis, viz. ex 48ℨ+192℞+64 aufer, restant +64, ex quibus aufer etiam cubum novissimi quotientis, qui est +64, et nihil restat, ut sequitur :—

+12ℨℨ+ 96ℭ+ 48ℨ
1ℨℭ+12ß +60ℨℨ+160ℭ+240ℨ+192℞+64 (1ℨ+4℞+4
3ℨℨ+48ℨℨ+ 64ℭ pro radice
+ 3ℨℨ+ 24ℭ+ 48ℨ cubicâ verâ.
+ 48ℨ+192℞+64

Aliud exemplum.

Sit sequentis compositi extrahenda radix cubica, qui sic primò collocetur :—

1ℭ—10ℨ+31℞—30 (1℞

Secundò, sic disponatur :—

— $\frac{7}{3}$℞+ $\frac{190}{27}$
1ℭ—10ℨ+ 31℞— 30 (1℞—3$\frac{1}{3}$ pro radice cubicâ proximâ,
+ 3ℨ+$\frac{100}{3}$℞—$\frac{1000}{27}$ non autem verâ, propter reliquias extantes.

3. Si compositi radicem veram extrahere volueris, qui tamen radicem veram insitam non habuerit, sed proximam tantum, ei composito præpone signum universale, et fiet inde radix obscura vera.

Ut ex secundo exemplo quadratorum præfato, viz. ex √ℭ4ℨ —√ℭ16℞—8 sit extrahenda radix vera quadrata, ea erit √ℨ. √ℭ4ℨ—√ℭ16℞—8.

Item, sit ex 1𝔠—10℥+31℞—30 extrahenda radix cubica vera, ea erit √𝔠. 1𝔠—10℥+31℞—30.

4. Radices autem quadrati quadratas, supersolidas, et cæteras superiores, tùm quia rarissimi sunt usus, tùm quia ex dictis considerari possunt, omittimus.

Ut si sit extrahenda radix quadrati quadrata, eam per regulam cubi sic emendatam extrahe. Primò, pro 'radicem cubicam extrahe,' lege, 'radicem quadrati quadratam extrahe.' Secundò, pro 'tria quadrata,' lege, 'quatuor cubos.' Tertiò, pro 'tria quadrata ducta in primum antecedentem quotientem,' lege, 'sex quadrata ducta in quadratum primi antecedentis quotientis et quatuor cubos novi ductos in primum antecedentem quotientem,' etc. Quartò, pro 'cubum novi,' lege, 'quadrati quadratum novi.' Et sic emendata regula ad quadrati quadratam radicem extrahendam inserviet.

At verò si pro supersolidâ radice extrahendâ regulam emendare volueris, pro 'cubicam,' lege, 'supersolidam,' et pro 'tria quadrata,' lege, 'quinque quadrati quadrata,' et pro 'tria quadrata ducta in primum antecedentem quotientem,' lege, 'decem quadrata ducta in cubum primi antecedentis quotientis, et decem cubos novi ductos in quadratum antecedentis, et quinque quadrati quadrata novi in primum antecedentem quotientem,' etc. Et pro 'cubum novi,' lege, 'supersolidum novi;' et simili modo, ad omnes superiores radices extrahendas, poterint constitui regulæ.

Exemplum regulæ quadrati quadratæ.

$$\underline{1a\text{℥℥}}+\underline{4a\mathfrak{C}b}+\underline{6a\text{℥}b\text{℥}}+\underline{4b\mathfrak{C}a}+\underline{1b\text{℥℥}} \quad (\ 1a+1b \text{ pro verâ radice}$$
$$+\underline{4a\mathfrak{C}}\ +\underline{6b\text{℥}a\text{℥}}+\underline{4b\mathfrak{C}a}+\underline{1b\text{℥℥}} \qquad \text{cubicâ.}$$

Exemplum regulæ supersolidæ.

1*a*ß+5*a*ʓʓ*b*+10*a*ℭ*b*ʓ+10ʓ*b*ℭ +5*b*ʓʓ*a*+1*b*ß (1*a*+1*b* vera radix
5*a*ʓʓ +10*b*ʓ*a*ℭ+10*b*ℭ*a*ʓ+5*b*ʓʓ*a*+1*b*ß supersolida.

5. Patet itaque ex præmissis quod aliquæ extractionum reliquiæ nulla habent signa positiva, et hæ reliquiæ totæ formales dicuntur; aliæ reliquiæ habent, et hæ totæ informales dicuntur.

Ut in exemplis 2, 3, et 4 extractionis quadratæ, reliquiæ omnes priorum operum sunt informales, eorundem autem exemplorum novissimæ reliquiæ sunt numerus; atque ideo formales dicuntur, Exemplis verò 5 et 6 quadratæ, et 2 cubicæ reliquiæ omnes, tùm primæ tùm novissimæ, eo quod positivis scatent, informales dicuntur.

Informatium reliquiarum quædam sunt formabiles, quædam reformabiles, quædam prorsus deformes et irreformabiles.

Ut exemplis proxime subsequentibus patebit.

6. Formabiles sunt reliquiæ cum quibus secunda pars regulæ extractionis exerceri possit, reliquias inde nullas, aut prioribus minus informales reddentes. Ipsumque opus secundæ partis regulæ extractionis Conformatio dicitur.

Ut in exemplis omnibus superioribus, et quadrati et cubi, reliquiæ omnes præter novissimas dicuntur formabiles, quia per solam secundam partem regulæ extractionis conformantur, et reliquiæ novissimæ inde minus informales exsurgent.

7. Reformabiles sunt reliquiæ quas si diviseris per compositum aliquod æquale nihilo (seu per æquationem ad 0), et hinc extantes

recentiores reliquias per aliam atque aliam ad 0 æquationem, si opus sit, diviseris; extabunt tandem reliquiæ aut nullæ, aut formales, aut formabiles, illæque æquationes Reformatrices vocabuntur, et ipsum opus dividendi Reformatio dicetur.

Exemplum.

Ex 1♀—0℞*a*+1*a*♀—1℞+1*a*—18 extrahatur radix quadrata proxima, (quod supra, exemplo 6, fit,) et erit radix 1℞—1*a*—$\frac{1}{2}$, et reliquiæ erunt informales, viz. +2℞*a*—18$\frac{1}{4}$; detur autem, exempli gratiâ, compositum hoc 1℞*a*+1℞—1*a*—10, quod nihilo æquetur; per hoc divide illas reliquias, et exsurgent reliquiæ —2℞+2*a*+1$\frac{3}{4}$, quæ, quia per prop. 6 hujus sunt formabiles; ideo reliquiæ +2℞*a*—18$\frac{1}{4}$ dicuntur reformabiles, et compositum 1℞*a*+1℞—1*a*—10 reformatrix, et opus ipsum reformatio dicentur.

Aliud exemplum.

Item ex 1♀+4℞*a*+1*a*♀—4℞*b*—4*ab*+4*b*♀+4℞+4*a*—8*b*—61 extrahatur radix quadrata, eaque erit 1℞+1*a*–2*b*+2, et reliquiæ erunt +2℞*a*—65 informales. Detur autem æquatio ad 0, quæ sit 1℞*a*—1*ab*—1*b*—5, per quam divide illas reliquias, et exsurgent reliquiæ +2*ab*+2*b*—55, quæ, quia nec formales nec formabiles sunt, debent per aliam æquationem ad 0 dividi, exempli gratiâ, per 2*ab*—3℞—3*a*+8*b*—21, et exsurgent reliquiæ 3℞+3*a*—6*b*—34, quæ, quia formabiles sunt (respectu, viz. præfatæ radicis proximæ, viz. 1℞+1*a*—2*b*+2), ideo et reliquiæ 2℞*a*—65, et reliquiæ 2*ab*+2*b*—55 reformabiles dicuntur, atque et compositum 1℞*a*—1*ab*—1*b*—5, et compositum 2*ab*—3℞—3*a*+8*b*—21 sunt reformatrices æquationes.

8. Ut igitur reliquiæ informales fiant formales, conformabiles conformabis (per 6 prop. hujus); et reformabiles (per 7,) reformabis, et reliquias omnium novissimas notabis, quæ si aut nullæ aut formales fuerint, bene est, tunc enim quotientes omnes conformationum copulandæ et abbreviandæ sunt, et erunt radix proxima reformata; quotientes verò reformationum inutiles semper, et spernendæ sunt.

Ut exempli penultimi radix quadrata proxima erat 1℞—1a—$\frac{1}{2}$, et reliquiæ +2℞a—18$\frac{1}{4}$, quas, quia reformabiles sunt (per prop. 7) per reformatricem suam 1℞a+1℞—1a—10 reformato, et spreto quotiente exsurgent reliquiæ —2℞+2a+1$\frac{3}{4}$, quæ quia formabiles sunt (per 7) conformato, et exsurgent reliquiæ formales notandæ, viz. +$\frac{3}{4}$ et quotiens conformationis —1 cum radice præfatâ copulatus et abbreviatus fiet 1℞—1a—1$\frac{1}{2}$ pro radice proximâ reformatâ.

Item exempli ultimi reliquiæ erant primò +2℞a—65, quas per suam reformatricem, viz. 1℞a—1ab—1b—5 reformato, et spreto quotiente exsurgent reliquiæ 2ab+2b—55 ut prop. 7 diximus; quæ rursus reformato per aliam reformatricem (ut ibidem monuimus,) viz. per 2ab—3℞—3a+8b—21, exsurgent spreto quotiente reliquiæ 3℞+3a—6b—34, quas, quia formabiles sunt (per 6 prop.) ad suam radicem proximam, viz. ad 1℞+1a —2b+2, conformabis, et erit tota radix proxima reformata 1℞ +1a—2b+3$\frac{1}{2}$, et reliquiæ novissimæ notandæ sunt —$\frac{169}{4}$ sive 42$\frac{1}{4}$ formales.

9. At si defectu reformatricium post ultimam conformationem exstent reliquiæ informales, hæ deformes aut irreformabiles appellantur.

Ut, novissimæ reliquiæ exempli 5 et 6 quadratæ, et 2 cubicæ extractionis (si nullæ occurrant reformatrices,) dicuntur deformes et irreformabiles.

10. Deformium reliquiarum et suarum radicum duæ sunt species, singulares et plurales, quarum singulares sunt eæ deformes quæ habent unum aliquod simplex et purum positivum, aut mixti positivi particulam unam in radice, seu quotiente cui non fuerit aliud simile vel ejusdem positionis, nec in quotiente seu radice nec inter reliquias.

Ut radix quadrata hujus 1ℨ+6℞*a*—7 est 1℞+3*a*, et reliquiæ sunt —9*a*ℨ—7, quæ ideò singulares dicuntur quia in eis non sunt plures positivi primæ positionis uno, qui est 1℞. Item radix quadrata hujus 1ℨ*a*ℨ—6℞*a*—1*a*+8=0 est 1℞*a*—3, et reliquiæ sunt —1*a*—1, in quibus signum ℞ non sæpius quàm semel reperitur.

11. Plurales dicuntur radices suæque reliquiæ, quum uniuscujusque positionis plures simplices in radice vel inter reliquias reperiuntur.

Ut radix cubica proxima hujus 1ℭ—9ℨ+36℞—80=0 est 1℞—3, et reliquiæ erunt +9℞—53. In quibus primæ positionis signum ℞ bis reperitur.

Item radix quadrata proxima hujus 1ℨ+1*a*ℨ—1℞+1*a*—18 (deficiente reformatrice,) erit 1℞—1*a*—$\frac{1}{2}$, et reliquiæ erunt 2℞*a*—18$\frac{1}{4}$, ut supra exemplo 6 docuimus, in quibus primæ positionis duæ sunt simplices, viz. 1℞ et 2℞*a*; item secundæ positionis totidem, viz. 1*a* et 2℞*a*.

12. Sunt itaque radicum quatuor formæ. Prima est verarum radicum, secunda est formalium, tertia singularium, quarta pluralium; quarum extrahendarum usum inferius docebimus.

CAPUT IX.

DE ÆQUATIONIBUS ET SUIS EXPONENTIBUS.

1. ÆQUATIO est positivorum incertorum valorum cum aliis sibi æqualibus collatio, ex quâ positionis valor quæritur.

Ut si quis pro numero quæsito aut quantitate quæsitâ ponens 1℞, ejus valorem ignorans, postea per hypothesin quæstionis deprehendens 3℞ æquari ad 21, conserat tres res cum suis æqualibus 21, ea æqualitatis collatio dicitur æquatio; et hinc infertur rem unam seu unam positionem valere 7.

2. Inter æquationis partes invicem æquales interjicitur linea duplex, quæ signum æquationis dicitur.

Ut 3℞=21, quæ sic pronuntiantur, tres res æquales viginti uni. Item 1℞=7, quæ pronuntiantur, una res æqualis ad septem.

3. Æquationum aliæ unius tantum sunt positionis, aliæ plurium positionum.

Unius tantum positionis, ut 1*a*ʓ+3*a*=10: Plurium positionum, ut 2ʓ—1*a*=6.

4. Item æquationum aliæ rudes, quæ ad minores terminos, magisque perspicuous et succinctos reduci possunt, aliæ perfectissimæ dicuntur, quæ è contra sunt maxime perspicuæ et succinctæ.

Ut, 3℞=21 est æquatio rudis, quia in perfectissimam, viz. in 1℞=7 reduci possit. Item, 5*a*ʓ=20 est æquatio rudis, quia in

perfectiorem, viz. in 1*a*ʒ=4 reduci possit. Sed et 1*a*ʒ=4 rudis est, quia adhuc in perfectiorem, imo perfectissimam, viz. in 1*a*=2 reduci possit, arte quam inferius tractabimus. Item 12ʒ +3*a*=6 rudis est æquatio, quia in perfectiorem 4ʒ+1*a*=2 reduci possit.

5. Item æquationum aliæ simplices, aliæ quadratæ, aliæ cubicæ, aliæ superiores: quarum simplices sunt quæ duobus ordinibus tantum constant.

Ut 3℞=27, seu 1℞=9; item 5*b*ʒ=20, simplices æquationes dicuntur.

6. Simplicium æquationum aliæ sunt reales, quæ sunt rerum æqualium ad numerum; aliæ radicales, quæ sunt quorundam quadratorum, cuborum, vel aliorum superiorum ad numerum æquationes.

Reales, ut 3℞=21, seu 1℞=7. Item 1*a*=3. Item 2℞=√ʒ3—1. Radicales, ut 2ʒ=8. Item 3ꝯ=24. Item 1*aß*=√ꝯ9, etc.

7. Æquatio quadrata est quæ tribus proportionalibus ordinibus constat.

Ut 2ʒ+3℞=4, seu 3℞=2ʒ—4. Item 1*a*ʒꝯ—10=3*a*ʒ. Item 12—√ʒ1℞=1℞.

8. Æquatio cubica est quæ quatuor proportionalibus ordinibus constat.

Ut 1ꝯ—9ʒ=24—26℞. Item 1ꝯ+0ʒ—2℞=4. Item 1*a*ʒꝯ—2*a*ʒ=4 est æquatio cubica, quia (per prop. 4 cap. 6) sic collocata 1*a*ʒꝯ+0*a*ʒʒ—2*a*ʒ=4, constat quatuor ordinibus.

9. Æquatio quadrati quadrata est quæ quinque; supersolida, quæ sex; quadrati cubica, quæ septem, proportionalibus ordinibus constant: Et sic de reliquis superioribus in infinitum.

Quadrati quadrata, ut 2ℨℨ—28ℭ+142ℨ=308℞—240. Supersolida, ut 1*b*ℨß—4*b*ℨℨℨ+1*b*ℨℭ—3*b*ℨℨ—1*b*ℨ=12. Quadrati cubica, ut 1*a*ℨℭ—3*a*ß+2*a*ℨℨ—6*a*ℭ+1*a*ℨ=1*a*+6.

10. Æquatio illusiva est ea quæ impossibile asserit, et siquis impossibile quærit in æquationem illusivam cadet ejus responsum.

Ut 1℞=3℞ est æquatio illudens, siquidem impossibile est quicquam suo triplo æquari. Item 1ℨ=4℞—5 est æquatio illudens, siquidem nullum quadratum possit æquari quatuor rebus seu radicibus suis, ablatis quinque; ut inferius patebit.

11. Expositio est reductio rudis æquationis ad perfectissimam et realem æquationem, et pars æquationis realis quæ uni rei æquatur dicitur Exponens, eaque quæstionem solvit.

Ut quum hæc rudis 3℞=21 reducitur ad hanc perfectissimam 1℞=7, exponens utriusque æquationis erit 7, quia uni rei (viz. ad 1℞) æquatur. Item hæc rudis 5ℨ=20 reducitur ad hanc perfectiorem 1ℨ=4, deinde ad hanc perfectissimam et realem 1℞=2, quod quidem opus reductionis dicitur expositio, et 2 exponens, quia uni rei æquatur: quæstionem verò exponente solvi postea docebimus.

12. Omnis æquatio præter illusivam habet saltem unicum exponens, validum sive invalidum.

Hoc postea docebimus, hîc præmonuisse sufficit.

13. Exponentia valida sunt ea quæ per se posita copulâ + notantur,

et semper sunt majora nihilo. Exponentia verò invalida sunt quæ per se posita copulâ — notantur, et hæc minora sunt nihilo.

Ut in hâc æquatione 1℞=7, septem sint exponens validum, quia (per prop. 1 cap. 6 Lib. I.) copulâ + notari subintelligitur. At in hâc reali æquatione 1℞=—7, exponens contrariâ ratione invalidum dicitur, quia copulâ — notatur sic, —7, estque nihilo minus.

14. Exponentium alia et numero solo et quantitate solâ, alia tantum numero solo, alia tantum quantitate solâ, alia partim hâc partim illo, alia neutro, exprimi possunt.

De his, suisque exemplis, latius per ordinem, capitibus 11, 12, 13, dicetur.

15. Omnis æquationis portio ditioni unius anterioris copulæ subdita Minima dicitur, quotcunque copulas et terminos habuerit; copulaque anterior et prædominans dicitur Ductrix; cæteræ verò copulæ Intermediæ dicuntur.

Ut in hâc æquatione 1ℭ—3+√ʒ2+$\frac{3℞—4}{1ʒ+1}$—√ʒ.6+√ʒ1℞=0, in quâ 1ℭ dicitur minima, et + dicitur ejus ductrix copula. Item 3 dicitur minima, et — ejus ductrix. Item √ʒ2 dicitur minima, et + ejus ductrix. Item $\frac{3℞—4}{1ʒ+1}$ dicitur minima, et + ejus ductrix, quia in totam fractionem extenditur ejus vis. Cæteræ verò copulæ hujus fractionis intermediæ dicuntur. Item √ʒ.6 +√ʒ1℞ dicitur minima, et copula — ejus ductrix, quia in aggregatum valorem totius universalis radicis extenditur ejus vis, et reliqua copula + intermedia dicitur.

CAPUT X.

DE GENERALI ÆQUATIONUM PRÆPARATIONE.

1. Præparatio est reductio æquationum rudium ad perfectiores, quas postea ad perfectissimas reales reducit expositio.

Ut *5a*♀=20 prius præparantur, et fient 1*a*♀=4: deinde exponuntur, et fient 1*a*=2. Quibus modis præparantur jam dicetur; quibus vero exponuntur postea patebit.

2. Præparantur et perspicuæ redduntur æquationes rudes quinque modis; transpositione, abbreviatione, divisione, multiplicatione, et extractione.

Quorum modorum regulæ et exempla sequuntur.

3. Si minimam ex unâ parte æquationis in contrariam transferas, illique contrariam copulam ductricem præposueris, erunt partes (ut antea) æquales, et Transpositio dicitur.

Ut, ex hujus æquationis 4℞—6=5℞—20 posteriore parte, si transposueris —20 in priorem partem æquationis, copulâ ejus mutatâ, hoc situ, 4℞—6+20=5℞: Item adhuc si transposueris 4℞, fient —4℞ hoc situ, —6+20=5℞—4℞. Item æquationis hujus 1♀—√♀.3♀—2=3*a*, si transposueris —√♀.3♀—2, erit +√♀.3♀—2, hoc situ, 1♀=3*a*+√♀.3♀—2; et si adhuc etiam transposueris 3*a*, erunt —3*a*, hoc situ, +♀—3*a*=√♀.3♀—2, et partes oppositæ æquales sunt, ut antea fuerant.

4. Si omnes minimas alterius partis æquationis (per præmissam) in contrariam partem transponas, totum compositum æquabitur nihilo, et

dicitur æquatio ad nihil; debetque hæc æquatio (per prop. 4 cap. 2 hujus) abbreviari.

Ut, in exemplo suprascripto 4℞—6=5℞—20, transpone 5℞—20, eruntque —5℞+20, hoc situ, 4℞—6—5℞+20=0, quæ abbreviata efficiunt —1℞+14=0, quæ æquatio ad nihil est. Item 1♀—√♀.3♀—2=3*a*, cujus partem sinistram si dextrorsum transponas, fiet 0=—1♀+√♀.3♀—2+3*a*, quæ quidem dicitur æquatio ad nihil.

5. Si positivus maximus à fronte habeat copulam —, converte omnes omnium minimarum ductrices, et producetur æquatio magis perspicua.

Exempla ut supra: Si —1℞+14 æquetur ad 0, hoc situ, —1℞+14=0, per consequens +1℞—14 æquabitur etiam ad 0, hoc situ, 1℞—14=0. Item, eodem modo ex —1♀+3*a*+√♀. 3♀—2=0 fiet 1♀—3*a*—√♀.3♀—2=0. Item ex hâc —1℞—1 +$\frac{32}{1a+1}$=0 fiet hæc, 1℞+1—$\frac{32}{1a+1}$=0.

6. Si æquationis positivos omnes maximi ordinis per unitatem signatam signis positivis et radicalibus ejusdem ordinis diviseris, et per quotientem diviseris totam æquationem; hinc producetur æquatio perspicua habens maximum ordinem unitate notatum.

Exemplum æquationis, 2ℭ—8♀+6℞=0: Divide positivum maximi ordinis, viz. 2ℭ per 1ℭ fiet quotiens 2; per duo igitur divide totam æquationem, fientque 1ℭ—4♀+3℞=0.

Item hujus æquationis, 3℞—√♀2♀—6=0, positivi maximi ordinis sunt 3℞—√♀2♀, qui quidem (per prop. 5 cap. 4 hujus) sunt ejusdem ordinis potentiæ, eorumque ordo est rerum; divide ergo 3R—√♀2♀ per 1℞, sive (quod idem est) per √♀1♀, fiet quotiens 3—√♀2; per hunc quotientem (per prop. 2 cap. 11 Lib. I.) divide totam æquationem, fietque hæc æquatio 1℞—$\frac{18}{3}$

$-\frac{\sqrt{\text{ℨ}}72}{7}=0$, quæ quamvis fractio sit, tamen magis perspicua est quàm prius, eo quod signum ℨ aufertur.

Item tertium exemplum, $1\text{℞}a+1a+1\text{℞}-31=0$, in quo sit tibi animo expurgare et delere signum mixtum, viz. $1\text{℞}a$: Divide ergo $1\text{℞}a+1a$ per $1a$, vel $1\text{℞}a+1\text{℞}$ per 1℞ (utram volueris loco maximi ordinis acceptare,) exempli gratiâ acceptetur 1℞ : divisâ itaque $1\text{℞}a+1\text{℞}$ per 1℞, orietur quotiens $1a+1$, per quem divide totam æquationem $1\text{℞}a+1\text{℞}+1a-31=0$, fiet æquatio hæc, $1\text{℞}+1-\frac{32}{1a+1}=0$, quæ licet fracta, tamen magis perspicua est quàm prius, eo quod signum mixtum quod prius obscurum erat jam aufertur.

7. Si æquationis minimus ordo fuerit positivus, tunc per unitatem signatam signis minimi ordinis divide totam æquationem, et proveniet inde æquatio perspicua habens numerum absolutum loco minimi ordinis.

Exemplum, $1\text{ℭ}-4\text{ℨ}+3\text{℞}=0$, quæ divide per unitatem minimi ordinis, viz. per 1℞, fiet $1\text{ℨ}-4\text{℞}+3=0$. Item $3\text{ℨ}-\sqrt{\text{ℨ}}2\text{℞}=0$, hæc divide per $\sqrt{\text{ℨ}}1\text{℞}$, fiet inde hæc æquatio $\sqrt{\text{ℨ}}9\text{ℭ}-\sqrt{\text{ℨ}}2=0$, quarum ultima series semper est numeri.

8. Si particulæ aliquæ æquationis sint veræ fractiones, inque earum denominatores duxeris totam æquationem, producetur æquatio integra, et plerumque magis perspicua.

Ut, in hâc $\frac{6\text{℞}-8\text{ℨ}}{1\text{ℭ}+3\text{℞}}+2=0$, sunt $\frac{6\text{℞}-8\text{ℨ}}{1\text{ℭ}+3\text{℞}}$, vere fractio licet abbreviabilis; duces ergo totam æquationem in denominatorem $1\text{ℭ}+3\text{℞}$, fientque $2\text{ℭ}+12\text{℞}-8\text{ℨ}=0$.

Item hanc æquationem $1\text{ℨ}+\frac{2\text{℞}}{3}-\frac{88}{75}=0$ duc per 3, fiet primò $3\text{ℨ}+2\text{℞}-\frac{264}{75}=0$, hæcque rursum per 75 duc, fientque $225\text{ℨ}+150\text{℞}-264=0$, quæ quidem æquationes integræ sunt, et expertes fractionum.

9. Si in æquatione fuerit radix universalis unica, eam à reliquo æquationis (per prop. 3) separabis, et utrumque æquationis latus in se duces toties quoties signum universale denotat, et producetur æquatio magis perspicua, nulla enim habebit universalia signa.

Exemplum, 2℥+3℞—√℥.12ℭ+4℥℥+18=0: Primò per transpositionem fiant 2℥+3℞=√℥.12ℭ+4℥℥+18; deinde latera quadrentur, quia signum universale est √℥, fientque 4℥℥+12ℭ+9℥=12ℭ+4℥℥+18, et per consequens transposita et abbreviata facient 1℥=2.

Aliud exemplum: √ℭ.2℞—6=3℞ ducantur cubice latera, fient 2℞—6=27ℭ, alias 2℞—27ℭ—6=0.

10. Si æquatio duabus radicibus universalibus consimiliter radicatis absque ullis aliis minimis constiterit, transpositione separentur, et in se multiplicentur quoties denotat universale signum; producitur æquatio perspicua nullius universalis radicis.

Ut √℥.2℞+5—√℥.3℞—4=0 separentur, et fient √℥.2℞+5=√℥.3℞—4; quadrate multiplicentur, et fient 2℞+5=3℞—4, et per transpositionem et abbreviationem 1℞—9=0.

11. Si æquatio constet duabus solis dissimiliter radicatis universalibus, separentur universalia, et ducatur utrumque latus in se ad utriusque signi universalis dissimilis qualitatem, proveniet æquatio perspicua sine universalibus.

Ut √ß.3℥+6—√℥.2℞—3=0; prius transpositione separentur, sic, √ß.3℥+6=√℥.2℞—3; deinde, latera in se quadrati supersolide ducantur, fientque 32ß—240℥℥+720ℭ—1080℥+810℞—243=9℥℥+36℥+36, quæ transposita et abbreviata fient 32ß—249℥℥+720ℭ—1116℥+810℞—279=0.

12. Si in æquatione fuerint duæ universales radices quadratæ, cum aliis quibusdam simplicibus aut uninomiis, universales ambas copulatas à cæteris separa, et utrumque latus in se quadrate duc, et fiet æquatio constans unicâ tantum universali radice, delendâ etiam per prop. 9 hujus.

Ut hæc æquatio $\frac{1}{2}+\surd\mathrm{Q}.\underline{48\frac{1}{4}+1\mathrm{R}-1}\mathrm{Q}+\frac{1}{2}\mathrm{R}-\surd\mathrm{Q}.79-\frac{3}{4}\mathrm{Q}=0$ sic transponantur, $\surd\mathrm{Q}.79-\frac{3}{4}\mathrm{Q}-\surd\mathrm{Q}.\underline{48\frac{1}{4}+1\mathrm{R}-1}\mathrm{Q}=\frac{1}{2}\mathrm{R}+\frac{1}{2}$; deinde quadretur utrumque latus, fientque $127\frac{1}{4}+1\mathrm{R}-1\frac{3}{4}\mathrm{Q}-\surd\mathrm{Q}.15247+316\mathrm{R}-460\frac{3}{4}\mathrm{Q}-\surd\mathrm{C}+3\mathrm{QQ}=\frac{1}{4}\mathrm{Q}+\frac{1}{2}\mathrm{R}+\frac{1}{4}$ transpone et abbrevia, fietque æquatio, $\surd\mathrm{Q}.15247+316\mathrm{R}-460\frac{3}{4}\mathrm{Q}-3\mathrm{C}+3\mathrm{QQ}=127+\frac{1}{2}\mathrm{R}-2\mathrm{Q}$, quæ tandem (per prop. 9) fient $1\mathrm{QQ}+1\mathrm{C}-47\mathrm{Q}-189\mathrm{R}+882=0$.

13. Si æquatio constet tribus radicibus universalibus quadratis absque aliis minimis, duæ quadratæ à reliquâ per transpositionem separentur, lateraque quadrentur, et proveniet æquatio unius tantum universalis, per prop. 9 delendæ.

Ut sit æquatio $\surd\mathrm{Q}.3\mathrm{R}+2+\surd\mathrm{Q}.2\mathrm{R}-1-\surd\mathrm{Q}.4\mathrm{R}-2=0$ separentur sic, $\surd\mathrm{Q}.3\mathrm{R}-2+\surd\mathrm{Q}.2\mathrm{R}+1=\surd\mathrm{Q}.4\mathrm{R}+2$; quadrentur latera, et fient $5\mathrm{R}-1+\surd\mathrm{Q}.6\mathrm{Q}-1\mathrm{R}-2=4\mathrm{R}+2$; deinde fient per abbreviationem $\surd\mathrm{Q}.6\mathrm{Q}-1\mathrm{R}-2=3-1\mathrm{R}$; postea (per prop. 9) fient $6\mathrm{Q}-1\mathrm{R}-2=1\mathrm{Q}-6\mathrm{R}+9$, et tandem fient $5\mathrm{Q}+5\mathrm{R}-11=0$, alias $1\mathrm{Q}+1\mathrm{R}-2\frac{1}{5}=0$.

14. Si æquatio constet tribus universalibus quadratis, cum unico uninomio aut simplici; transponantur duæ universales à reliquis, lateraque quadrentur, et proveniet æquatio duarum universalium radicum, per prop. 12 delendarum.

Ut sit æquatio $\surd\mathrm{Q}.\underline{\surd\mathrm{C}2\mathrm{R}+3+\surd\mathrm{Q}.3\mathrm{R}-2}-2\mathrm{R}-\surd\mathrm{Q}.\underline{2\mathrm{Q}+1}=0$: transponantur sic, $\surd\mathrm{Q}.\surd\mathrm{C}2\mathrm{R}+3+\surd\mathrm{Q}.3\mathrm{R}-2=2\mathrm{R}+\surd\mathrm{Q}.2\mathrm{Q}+1$,

multiplicentur latera in se quadrate, fientque √ℨ.√ℭ3456ℨℨ —√ℭ1024℞+36℞—24+√ℭ2℞+3℞+1=6ℨ+1+√ℨ.32ℨℨ +8℞ constantia duobus universalibus quadratis, per prop. 12 delendis.

15. Si æquatio constet quatuor universalibus quadratis absque aliis minimis; binæ à binis per transpositionem separentur, lateraque quadrentur, et proveniet æquatio duarum tantum universalium, per prop. 12 delendarum.

Sit æquatio hæc sic transposita, √ℨ.5ℨ—2℞—√ℨ.10—1℞= √ℨ.2℞+6+√ℨ.1ℨ+4, cujus latera quadrentur, fientque 5ℨ —3℞+10—√ℨ.208ℨ-20ℭ-80℞=1ℨ+2℞+10+√ℨ.8ℭ+24ℨ +32℞+96, quæ duabus universalibus tantum constant, per prop. 12 delendis.

16. Si universalissima unica ex uno latere æquetur universalissimæ soli, sive universali soli, sive universali et uninomio aut simplici unicis, sive uninomiis et simplicibus tantum ex altero latere: tunc duc in se latera ad signorum universalium qualitates, et signa universalissima delebuntur, cæteris universalibus delebilibus per præcedentia delendis.

Ut in æquatione hâc √ℨ.10+√ℨ.5℞—2=√ℨ.3+√ℨ.3℞+1, universalissima universalissimæ æquatur; latera ergo quadrentur, fientque 10+√ℨ.5℞-2=3+√ℨ.3℞+1, sive 7+√ℨ.5℞-2 =√ℨ.3℞+1, quorum universalia per prop. 12 delebis.

Aliud exemplum.

Item hujus √ℨß.3+√ℨ.2℞—1=√ℭß.5+√ℨ.3℞—4 ducantur latera in se quadrati cubicè supersolidè, fientque 18℞ +18+√ℨ.8ℭ—12ℨ+6℞—1+√ℨ.1458℞—729=21+3℞ +√ℨ.300℞—400, sive 15℞—3+√ℨ.8ℭ—12ℨ+6℞—1

+√𝔮.1458℞—729=√𝔮.300℞—400, quorum universalia deleri nequiunt.

Tertium exemplum.

Item hujus √𝔠.3+√𝔮.2℞—1=√𝔠.20—4℞, duc cubice in se latera, fientque 3+√𝔮.2℞—1=20—4℞, sive √𝔮.2℞—1 =17—4℞, quorum universale (per prop. 9) delebis. Eadem est similium ratio.

17. Eisdem propositionibus quibus universales deleri dictum est, possunt et simplices irrationales inter rationales transponi, multiplicari, et tandem deleri.

Ut sit æquatio 12—√𝔮1℞=1℞, per prop. 9 separentur, sic, 12—1℞=√𝔮1℞, et multiplicentur quadrate latera, fientque 1𝔮—24℞+144=1℞, sive 1𝔮—25℞+144=0, quæ prorsus rationales sunt. Quæ itaque, propositionibus 9, 10, 11, 12, 13, 14, et 15, dicuntur de universalibus, eadem de simplicibus radicatis etiam dici intelligantur.

18. Quæ aliter præparari possunt æquationes, per propositionem ne præparentur præmissam; multiplicatio enim irrationalium simplicium plerumque plura exponentia debito exhibet.

Ut præcedens exemplum 12—√𝔮1℞=1℞, per præmissam multiplicatum, reddit æquationem 1𝔮—25℞+144=0, quæ duo habet valida exponentia, viz. 16 et 9, cum revera ipsa principalis æquatio, 12—√𝔮1℞=1℞, habeat unicum exponens tantum, viz. 9, ut postea patebit. Illa igitur æquatio principalis per prop. 17 ne præparetur, dummodo eadem per prop. 20 subsequentem melius et simplicius præparari possit, ut ibidem dicetur.

19. Si æquationis ad 0 extrahatur radix aliqua vera (viz. relicto nihilo), radix illa erit magis succincta, et ad 0 æquatio.

Ut ex æquatione 1𝔆—6ℨ+12℞—8=0 extrahe radicem cubicam veram, viz. 1℞—2=0, quæ erit abbreviata et succincta æquatio.

Item æquationis 1℞—√ℨ36℞+9=0 radicem quadratam extrahe, eaque erit vera (per cap. 8), viz. √ℨ1℞—3=0, quæ est magis succincta æquatio.

20. Si æquationis ad 0 extracta radix aliqua sit, aut formalis aut (per prop. 8 cap. 8 hujus) reformata; reliquiarum copulam converte, et earundem radices quadratas vel cubicas, etc. quales ex reliquo extrahe; has radices (conversis copulis) cum radice proximâ et formali copulato, fient æquationes, et unica, non quadratinomia vel duæ quadratinomiæ, ad 0 magis succinctæ, priorisque æquationis exponentia complectentes.

Et cætera.

Ther is no more of his algebra orderlie sett doun.

EDINBURGI: EXCUDEBANT BALLANTYNE ET HUGHES.